Connection between difference and sum labelings

Martin Bača,

Department of Appl. Mathematics, Technical University, Košice

45. Česko-slovenská konferencia GRAFY 2010

Lednice, 31. máj - 4. jún 2010

1. jún 2010

Joint work with

Christian Barrientos, University of Central Florida, Orlando, USA

Petr Kovář, Technical University of Ostrava, Czech Republic

Yuqing Lin, The University of Newcastle, Australia

Francesc A. Muntaner-Batle, Universitat Internacional de Catalunya, Barcelona, Spain

Andrea Semaničová - Feňovčíková, Technical University, Košice

Muhammad Kashif Shafiq, GC University, Lahore, Pakistan

Outline

(a, d)-edge-antimagic vertex labeling

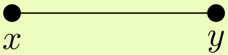
Definition

By an *(a, d)-edge-antimagic vertex ((a, d)–EAV) labeling* of a (p, q)-graph $G = (V, E)$ we mean a one-to-one mapping $f : V(G) \to \{1, 2, \ldots, p\}$ such that the set of edge-weights of all edges in G, $\{f(x) + f(y) : xy \in E(G)\}$ is $\{a, a + d, a + 2d, \ldots, a + (q - 1)d\}$, where $a > 0$ and $d \geq 0$ are two fixed integers.

(a, d)-*edge-antimagic vertex labeling*

Definition

By an ***(a, d)-edge-antimagic vertex ((a, d)–EAV) labeling*** of a (p, q)-graph $G = (V, E)$ we mean a one-to-one mapping $f : V(G) \to \{1, 2, \ldots, p\}$ such that the set of edge-weights of all edges in G, $\{f(x) + f(y) : xy \in E(G)\}$ is $\{a, a + d, a + 2d, \ldots, a + (q - 1)d\}$, where $a > 0$ and $d \geq 0$ are two fixed integers.

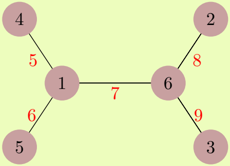


$$w_f(xy) = f(x) + f(y)$$

(a, d)-edge-antimagic vertex labeling

Definition

By an *(a, d)-edge-antimagic vertex ((a, d)–EAV) labeling* of a (p, q)-graph $G = (V, E)$ we mean a one-to-one mapping $f : V(G) \to \{1, 2, \ldots, p\}$ such that the set of edge-weights of all edges in G, $\{f(x) + f(y) : xy \in E(G)\}$ is $\{a, a + d, a + 2d, \ldots, a + (q - 1)d\}$, where $a > 0$ and $d \geq 0$ are two fixed integers.



$(5, 1)$–EAV labeling

(a, d)-edge-antimagic total labeling

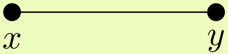
Definition

An *(a, d)-edge-antimagic total ((a, d)-EAT) labeling* of a (p, q)-graph $G = (V, E)$ is defined as a one-to-one mapping $f : V(G) \cup E(G) \to \{1, 2, \ldots, p + q\}$ so that the set of edge-weights $\{f(x) + f(xy) + f(y) : xy \in E(G)\}$ is equal to $\{a, a + d, a + 2d, \ldots, a + (q - 1)d\}$, for two integers $a > 0$ and $d \geq 0$.

(a, d)-edge-antimagic total labeling

Definition

An *(a, d)-edge-antimagic total ((a, d)-EAT) labeling* of a (p, q)-graph $G = (V, E)$ is defined as a one-to-one mapping $f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p + q\}$ so that the set of edge-weights $\{f(x) + f(xy) + f(y) : xy \in E(G)\}$ is equal to $\{a, a + d, a + 2d, \ldots, a + (q - 1)d\}$, for two integers $a > 0$ and $d \geq 0$.

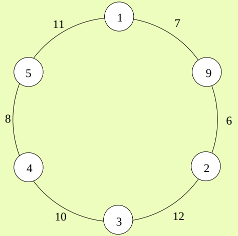


$$w_f(xy) = f(x) + f(xy) + f(y)$$

(a,d)-edge-antimagic total labeling

Definition

An *(a,d)-edge-antimagic total ((a,d)-EAT) labeling* of a (p,q)-graph $G=(V,E)$ is defined as a one-to-one mapping $f: V(G)\cup E(G)\to\{1,2,\dots,p+q\}$ so that the set of edge-weights $\{f(x)+f(xy)+f(y): xy\in E(G)\}$ is equal to $\{a, a+d, a+2d,\dots,a+(q-1)d\}$, for two integers $a>0$ and $d\geq 0$.



(a, d)-edge-antimagic total labeling

Definition

An *(a, d)-edge-antimagic total ((a, d)-EAT) labeling* of a (p, q)-graph $G = (V, E)$ is defined as a one-to-one mapping $f : V(G) \cup E(G) \to \{1, 2, \ldots, p + q\}$ so that the set of edge-weights $\{f(x) + f(xy) + f(y) : xy \in E(G)\}$ is equal to $\{a, a + d, a + 2d, \ldots, a + (q - 1)d\}$, for two integers $a > 0$ and $d \geq 0$.

• R. Simanjuntak, F. Bertault and M. Miller, *Two new (a, d)-antimagic graph labelings*, Proc. of Eleventh Australian Workshop of Combinatorial Algorithm (2000), 179-189.

(a, d)-edge-antimagic total labeling

Definition

An *(a, d)-edge-antimagic total ((a, d)-EAT) labeling* of a (p, q)-graph $G = (V, E)$ is defined as a one-to-one mapping $f : V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$ so that the set of edge-weights $\{f(x) + f(xy) + f(y) : xy \in E(G)\}$ is equal to $\{a, a+d, a+2d, \ldots, a+(q-1)d\}$, for two integers $a > 0$ and $d \geq 0$.

• R. Simanjuntak, F. Bertault and M. Miller, *Two new (a, d)-antimagic graph labelings*, Proc. of Eleventh Australian Workshop of Combinatorial Algorithm (2000), 179-189.

• A. Kotzig and A. Rosa, *Magic valuations of finite graphs*, Canad. Math. Bull. **13** (1970), 451–461.

Super (a, d)*-edge-antimagic total labeling*

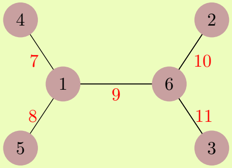
Definition

(a, d)-EAT labeling is called ***super*** if the vertex labels are the integers $\{1, 2, \ldots, p\}$.

Super (a, d)-edge-antimagic total labeling

Definition

(a, d)-EAT labeling is called ***super*** if the vertex labels are the integers $\{1, 2, \ldots, p\}$.

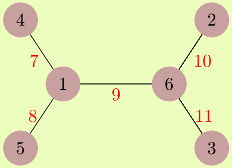


$w = \{12, 14, 16, 18, 20\}$
super $(12, 2)$–EAT labeling

Super (a, d)-edge-antimagic total labeling

Definition

(a, d)-EAT labeling is called ***super*** if the vertex labels are the integers $\{1, 2, \ldots, p\}$.



$$w = \{12, 14, 16, 18, 20\}$$
super $(12, 2)$–EAT labeling

• H. Enomoto, A.S. Lladó, T. Nakamigawa and G. Ringel, *Super edge-magic graphs*, SUT J. Math. **34** (1998), 105–109.

Conjectures

Kotzig - Rosa conjecture

Every tree is edge-magic.

Enomoto - Lladó - Nakamigawa - Ringel conjecture

Every tree is super edge-magic.

Upper bound of the difference d

Let (p,q)-graph be a super (a,d)-EAT.

The minimum possible edge-weight is at least $p+4$.
The maximum possible edge-weight is no more than $3p+q-1$.

$$a+(q-1)d \le 3p+q-1$$
$$d \le \frac{2p+q-5}{q-1}.$$

For any (p,q)-graph, where $p-1 \le q$, it follows that $d \le 3$

Super $(a, 1)$-EAT labeling of regular graphs

Theorem

Let G be a graph on p vertices that can be decomposed into two factors G_1 and G_2. If G_1 is edge-empty or if G_1 is a super $(2p + 2, 1)$-EAT graph and G_2 is a $2r$-regular graph then G is super $(2p + 2, 1)$-EAT.

• M. Bača, P. Kovář, A. Semaničová–Feňovčíková and M.K. Shafiq, *On super $(a, 1)$-edge-antimagic total labelings of regular graphs*, Discrete Math. **310** (2010), 1408–1412.

Super $(a, 1)$-EAT labeling of regular graphs

Theorem

Let G be a graph on p vertices that can be decomposed into two factors G_1 and G_2. If G_1 is edge-empty or if G_1 is a super $(2p+2, 1)$-EAT graph and G_2 is a $2r$-regular graph then G is super $(2p+2, 1)$-EAT.

Proposition

(Petersen Theorem) *Let G be a $2r$-regular graph. Then there exists a 2-factor in G.*

- M. Bača, P. Kovář, A. Semaničová–Feňovčíková and M.K. Shafiq, *On super $(a, 1)$-edge-antimagic total labelings of regular graphs*, Discrete Math. **310** (2010), 1408–1412.

Super $(a, 1)$-EAT labeling of regular graphs

Theorem

Let G be a graph on p vertices that can be decomposed into two factors G_1 and G_2. If G_1 is edge-empty or if G_1 is a super $(2p+2, 1)$-EAT graph and G_2 is a $2r$-regular graph then G is super $(2p+2, 1)$-EAT.

Proposition

(Petersen Theorem) *Let G be a $2r$-regular graph. Then there exists a 2-factor in G.*

Corollary

All even-regular graphs of order p with at least one edge are super $(2p+2, 1)$-EAT.

Odd-regular graphs with a perfect matching

Lemma

If G is a super $(a, 1)$-EAT graph then also $G \cup mK_1$ is a super $(a + m + 2t, 1)$-EAT graph for all $t \in \{0, 1, \ldots, m\}$.

Odd-regular graphs with a perfect matching

Lemma

If G is a super $(a, 1)$-EAT graph then also $G \cup mK_1$ is a super $(a + m + 2t, 1)$-EAT graph for all $t \in \{0, 1, \ldots, m\}$.

Lemma

Let k, m be positive integers. Then the graph $kP_2 \cup mK_1$ is super $(2(2k + m) + 2, 1)$-EAT.

• M. Bača, Y. Lin, M. Miller, R. Simanjuntak, *New constructions of magic and antimagic graph labelings*, Utilitas Math. **60** (2001), 229–239.

Odd-regular graphs with a perfect matching

Lemma

If G is a super $(a, 1)$-EAT graph then also $G \cup mK_1$ is a super $(a + m + 2t, 1)$-EAT graph for all $t \in \{0, 1, \ldots, m\}$.

Lemma

Let k, m be positive integers. Then the graph $kP_2 \cup mK_1$ is super $(2(2k + m) + 2, 1)$-EAT.

Theorem

If G is an odd regular graph on p vertices that has a 1-factor, then G is super $(2p + 2, 1)$-EAT.

Cartesian product $G \times K_2$

Corollary

Let G be a regular graph. Then the Cartesian product $G \times K_2$ is a super $(a, 1)$-EAT graph.

Generalized Petersen graph

Theorem

Every generalized Petersen graph $P(n, m)$, $n \geq 3$, $1 \leq m < \frac{n}{2}$, has a super $(4n + 2, 1)$-EAT labeling.

- A.A.G. Ngurah and E.T. Baskoro, *On magic and antimagic total labeling of generalized Petersen graph*, Utilitas Math. **63** (2003), 97–107.

Generalized Petersen graph

Theorem

Every generalized Petersen graph $P(n,m)$, $n \geq 3$, $1 \leq m < \frac{n}{2}$, has a super $(4n+2,1)$-EAT labeling.

- A.A.G. Ngurah and E.T. Baskoro, *On magic and antimagic total labeling of generalized Petersen graph*, Utilitas Math. **63** (2003), 97–107.

Theorem

The graph mK_n has a super (a,d)-EAT labeling if and only if either

(i) $d \in \{0,2\}$, $n \in \{2,3\}$ and m is odd, $m \geq 3$, or

(ii) $d = 1$, $n \geq 2$ and $m \geq 2$, or

(iii) $d \in \{3,5\}$, $n = 2$ and $m \geq 2$, or

(iv) $d = 4$, $n = 2$ and m is odd, $m \geq 3$.

- M. Bača and C. Barrientos, *On super edge-antimagic total labelings of mKn*, *Discrete Mathematics* **308**, No. 22 (2008), 5032–5037.

Union of cycles

Theorem

The graph mC_n has a super (a, d)-EAT labeling if and only if either
(i) $d \in \{0, 2\}$ and m, n are odd, $m, n \geq 3$; or
(ii) $d = 1$, for every $m \geq 2$ and $n \geq 3$.

- Dafik, M. Miller, J. Ryan and M. Bača, *On super (a,d)-edge-antimagic total labeling of disconnected graphs*, *Discrete Mathematics* **309** (2009), 4909–4915.

Non-regular super $(a, 1)$-EAT graphs

Lemma

Let k, m be positive integers, $k < 2m + 3$. Then the graph $K_{1,k} \cup mK_1$ is super $(2(k + m + 1) + 2, 1)$-EAT.

Non-regular super $(a, 1)$-EAT graphs

Lemma

Let k, m be positive integers, $k < 2m + 3$. Then the graph $K_{1,k} \cup mK_1$ is super $(2(k + m + 1) + 2, 1)$-EAT.

Lemma

Let k, m be positive integers, let m be even. Let H be an arbitrary 2-regular graph of order k. Then the graph $H \cup mK_1$ is super $(2(k + m) + 2, 1)$-EAT.

Non-regular super $(a, 1)$-EAT graphs

Lemma

Let k, m be positive integers, $k < 2m + 3$. Then the graph $K_{1,k} \cup mK_1$ is super $(2(k + m + 1) + 2, 1)$-EAT.

Lemma

Let k, m be positive integers, let m be even. Let H be an arbitrary 2-regular graph of order k. Then the graph $H \cup mK_1$ is super $(2(k + m) + 2, 1)$-EAT.

Lemma

Let k, m be positive integers, let m be even. Then the graph $P_k \cup mK_1$ is super $(2(k + m) + 2, 1)$-EAT.

Non-regular super $(a, 1)$-EAT graphs

Theorem

Let k, m be positive integers. Let G be a graph on p vertices that can be decomposed into two factors G_1 and G_2. If G_2 is a $2r$-regular graph and either

1) *G_1 is the graph $kP_2 \cup mK_1$, or*
2) *G_1 is the graph $K_{1,k} \cup mK_1$ for $k < 2m + 3$, or*
3) *H is an arbitrary 2-regular graph of order k and $G_1 \cong H \cup mK_1$ for even m, or*
4) *G_1 is the graph $P_k \cup mK_1$ for even m,*

then the graph G is super $(2p + 2, 1)$-EAT.

• M. Bača, P. Kovář, A. Semaničová–Feňovčíková and M.K. Shafiq, *On super $(a, 1)$-edge-antimagic total labelings of regular graphs*, Discrete Math. **310** (2010), 1408–1412.

$(a, 0)$-EAT labelings

Theorem

G is super $(a, 0)$-EAT, if and only if G admits an $(a, 1)$-EAV labeling.

- R.M. Figueroa-Centeno, R. Ichishima and F.A. Muntaner-Batle, *The place of super edge-magic labelings among other classes of labelings*, *Discrete Math.* **231** (2001) 153–168.

$(a, 0)$-*EAT labelings*

Theorem

G is super $(a, 0)$-EAT, if and only if G admits an $(a, 1)$-EAV labeling.

• R.M. Figueroa-Centeno, R. Ichishima and F.A. Muntaner-Batle, *The place of super edge-magic labelings among other classes of labelings*, *Discrete Math.* **231** (2001) 153–168.

Proposition

If (p, q)-graph G has an (a, d)-EAV labeling then

i) *G has a super $(a + p + 1, d + 1)$-EAT labeling,*

ii) *G has a super $(a + p + q, d - 1)$-EAT labeling.*

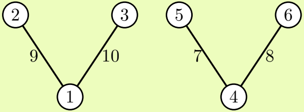
• M. Bača, Y. Lin, M. Miller, R. Simanjuntak, *New constructions of magic and antimagic graph labelings*, Utilitas Math. **60** (2001), 229–239.

$(a, 0)$-*EAT labelings*

The condition in Proposition is only sufficient for the existence of a super $(a, 2)$-EAT labeling from the existence of a super $(a, 0)$-EAT labeling of a graph.

$(a, 0)$-*EAT labelings*

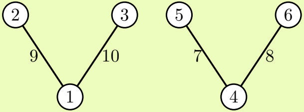
The condition in Proposition is only sufficient for the existence of a super $(a, 2)$-EAT labeling from the existence of a super $(a, 0)$-EAT labeling of a graph.



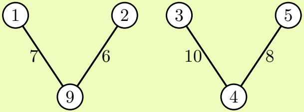
Super $(12, 2)$–EAT labeling of $2P_3$.

$(a, 0)$-*EAT labelings*

The condition in Proposition is only sufficient for the existence of a super $(a, 2)$-EAT labeling from the existence of a super $(a, 0)$-EAT labeling of a graph.



Super $(12, 2)$–EAT labeling of $2P_3$.



$(17, 0)$–EAT labeling of $2P_3$.

$(a, 2)$-EAT labelings

Proposition

If G is a (super) edge-magic bipartite or tripartite graph and m is odd, then mG is (super) edge-magic.

- R.M. Figueroa-Centeno, R. Ichishima and F.A. Muntaner-Batle, *On edge-magic labelings of certain disjoint unions of graphs*, *Australas. J. Combin.* **32** (2005) 225–242.

$(a, 2)$-EAT labelings

Proposition

If G is a (super) edge-magic bipartite or tripartite graph and m is odd, then mG is (super) edge-magic.

- R.M. Figueroa-Centeno, R. Ichishima and F.A. Muntaner-Batle, *On edge-magic labelings of certain disjoint unions of graphs*, *Australas. J. Combin.* **32** (2005) 225–242.

Theorem

If G is a (super) $(a, 2)$-EAT tripartite graph and m is odd, then mG is (super) $(a', 2)$-EAT.

- M. Bača, F.A. Muntaner-Batle, A. Semaničová–Feňovčíková and M.K. Shafiq, *On super $(a, 2)$-edge-antimagic total labeling of disconnected graphs*, *Ars Combin.*, in press.

$(a, 2)$-EAT labelings

Proposition

If G is a (super) edge-magic bipartite or tripartite graph and m is odd, then mG is (super) edge-magic.

- R.M. Figueroa-Centeno, R. Ichishima and F.A. Muntaner-Batle, *On edge-magic labelings of certain disjoint unions of graphs, Australas. J. Combin.* **32** (2005) 225–242.

Theorem

If G is a (super) $(a, 2)$-EAT tripartite graph and m is odd, then mG is (super) $(a', 2)$-EAT.

Corollary

If G is a (super) $(a, 2)$-EAT bipartite graph and m is odd, then mG is (super) $(a', 2)$-EAT.

$(a, 0)$-*EAT labelings*

A mapping $c : V(G) \cup E(G) \to \{1, 2, 3\}$ is called an *e-m-coloring* of a graph G if $\{c(u), c(v), c(uv)\} = \{1, 2, 3\}$ for any edge uv of G.

$(a, 0)$-*EAT labelings*

Proposition

Let m be an odd positive integer. For $i = 1, 2, \ldots, m$, let G_i, g_i and c_i be an edge-magic graph with p_i vertices and q_i edges, an edge-magic total labeling of G_i with its magic number σ_i and an e-m-coloring of G_i, respectively. Suppose that the following conditions are satisfied

1. *there is an integer σ such that $\sigma_i = \sigma$ for all $i = 1, 2, \ldots, m$,*
2. *if $g_i(x) = g_j(y)$, then $c_i(x) = c_j(y)$ for all $i, j = 1, 2, \ldots, m$, $x \in V(G_i) \cup E(G_i)$ and $y \in V(G_j) \cup E(G_j)$,*
3. *there is an integer r such that* $r = p_1 + q_1 \geq \cdots \geq p_m + q_m \geq r - 1$.

Then the disjoint union $\cup_{i=1}^m G_i$ is an edge-magic graph.
Moreover, if all g_i are super edge-magic labelings and $p_1 = p_2 = \cdots = p_m$, then $\cup_{i=1}^m G_i$ is a super edge-magic graph.

- J. Ivančo and I. Lučkaničová, *On edge-magic disconnected graphs, SUT J. Math.* **38** (2002) 175–184.

$(a,2)$-EAT labelings

Theorem

Let m be an odd positive integer. For $i = 1, 2, \ldots, m$, let G_i, f_i and c_i be an $(a,2)$-EAT graph with p_i vertices and q_i edges, an $(a,2)$-EAT labeling of G_i and an e-m-coloring of G_i, respectively. Suppose that the following conditions are satisfied

1. *if $f_i(x) = f_j(y)$, then $c_i(x) = c_j(y)$ for all $i, j = 1, 2, \ldots, m$, $x \in V(G_i) \cup E(G_i)$ and $y \in V(G_j) \cup E(G_j)$,*
2. *there is an integer r such that $r = p_1 + q_1 \geq \cdots \geq p_m + q_m \geq r - 1$.*

Then the disjoint union $\cup_{i=1}^m G_i$ is an $(a',2)$-EAT graph. Moreover, if all f_i are super $(a,2)$-EAT labelings and $p_1 = p_2 = \cdots = p_m$, then $\cup_{i=1}^m G_i$ is a super $(a',2)$-EAT graph.

• M. Bača, F.A. Muntaner-Batle, A. Semaničová–Feňovčíková and M.K. Shafiq, *On super $(a,2)$-edge-antimagic total labeling of disconnected graphs*, *Ars Combin.*, in press.

Non-existence of (a, d)-EAT labelings

Proposition

Let G be a graph with all vertices of odd degrees. If $|E(G)| \equiv 0 \pmod 2$ and $|V(G)| + |E(G)| \equiv 2 \pmod 4$ then G has no $(a, 0)$-EAT labeling.

• G. Ringel and A. Lladó, *Another tree conjecture*, Bull. Inst. Combin. Appl. **18** (1996), 83–85.

Non-existence of (a, d)-EAT labelings

Proposition

Let G be a graph with all vertices of odd degrees. If $|E(G)| \equiv 0 \pmod 2$ and $|V(G)| + |E(G)| \equiv 2 \pmod 4$ then G has no $(a, 0)$-EAT labeling.

- G. Ringel and A. Lladó, *Another tree conjecture*, Bull. Inst. Combin. Appl. **18** (1996), 83–85.

Theorem

Let G be a graph with all vertices of odd degrees and let d be an even integer. If one of the following conditions holds

i) $|E(G)| \equiv 0 \pmod 4$ *and* $|V(G)| \equiv 1 \pmod 4$ *or* $|V(G)| \equiv 2 \pmod 4$,

ii) $|E(G)| \equiv 2 \pmod 4$ *and* $|V(G)| \equiv 0 \pmod 4$ *or* $|V(G)| \equiv 3 \pmod 4$

then G has no (a, d)-EAT labeling.

Non-existence of (a, d)-EAT labelings

Theorem

Let d, k be positive integers, d even and k odd. Let G be a graph with all vertices of odd degrees. If the size and the order of G have a different parity then the graph $2kG$ has no (a, d)-EAT labeling.

Graceful labeling

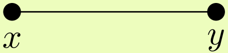
Definition

A ***graceful labeling (β-valuation)*** of graph $G = (V, E)$ is an injection $f : V(G) \to \{1, 2, \ldots, |E(G)| + 1\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct.

Graceful labeling

Definition

A ***graceful labeling (β-valuation)*** of graph $G = (V, E)$ is an injection $f : V(G) \to \{1, 2, \ldots, |E(G)| + 1\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct.

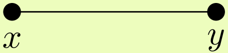


$$wt_f(xy) = |f(x) - f(y)|$$

Graceful labeling

Definition

A ***graceful labeling (β-valuation)*** of graph $G = (V, E)$ is an injection $f : V(G) \to \{1, 2, \ldots, |E(G)| + 1\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct.



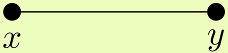
$$wt_f(xy) = |f(x) - f(y)|$$

- A. Rosa, *On certain valuations of the vertices of a graph*, Internat. Symposium, Rome, 1966.

Graceful labeling

Definition

A ***graceful labeling (β-valuation)*** of graph $G = (V, E)$ is an injection $f : V(G) \to \{1, 2, \ldots, |E(G)| + 1\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct.



$$wt_f(xy) = |f(x) - f(y)|$$

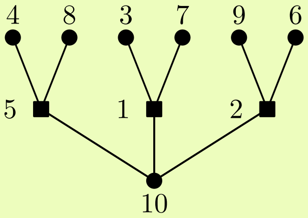
• A. Rosa, *On certain valuations of the vertices of a graph*, Internat. Symposium, Rome, 1966.

• S.W. Golomb, *How to number a graph*, in Graph Theory and Computing, Academic Press, New York (1972), 23-37.

Graceful labeling

Definition

A ***graceful labeling (β-valuation)*** of graph $G = (V, E)$ is an injection $f : V(G) \rightarrow \{1, 2, \ldots, |E(G)| + 1\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct.



Graceful labeling of symmetrical tree.

Graceful graphs

Ringel - Kotzig conjecture

All trees are graceful.

Graceful graphs

Ringel - Kotzig conjecture

All trees are graceful.

Among the trees known to be graceful are

- caterpillars (A. Rosa, 1966)
- trees with at most 4 end-vertices (C. Huang, A. Kotzig, A. Rosa, 1982)
- trees with diameter at most 5 (P. Hrnčiar, A. Haviar, 2001)
- trees with at most 27 vertices (R.E.L. Aldred, B.D. McKay, 2007)
- lobsters (J.G. Wang, D.J. Jin, X.G. Lu and D. Zhang, 1994)
  (W.C. Chen, H.I. Lu and Y.N. Yeh, 1997)
  (D. Mishra and P. Panigrahi, 2006,2007)

Graceful graphs

Among the trees known to be graceful are

- symmetrical trees (J.C. Bermond and D. Sotteau, 1976)
  (S. Poljak and M. Sûra, 1982)
- banana trees (V. Bhat-Nayak and U. Deshmukh, 1996)
  (M. Murugan and G. Arumugan, 2001)

α-labeling

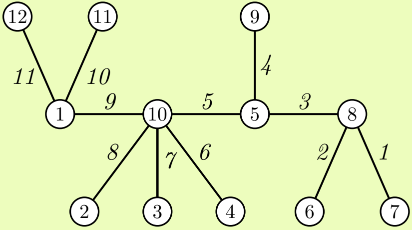
Definition

A graceful labeling of a graph G is said to be ***α-labeling*** if there exists an integer λ such that for each edge xy of G either $f(x) \leq \lambda < f(y)$ or $f(y) \leq \lambda < f(x)$.

α-labeling

Definition

A graceful labeling of a graph G is said to be *α-labeling* if there exists an integer λ such that for each edge xy of G either $f(x) \leq \lambda < f(y)$ or $f(y) \leq \lambda < f(x)$.



α-labeling of a caterpillar
$\lambda = 7$

α-labeling

An α-graph is necessary bipartite and for any of its α-labelings the vertices whose labels do not exceed the value λ form one of the sets of the bipartition.

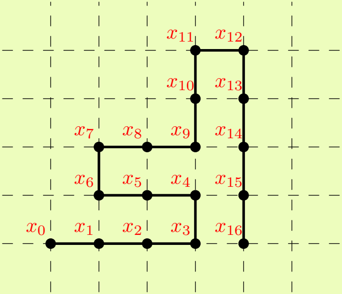
α-labeling

An α-graph is necessary bipartite and for any of its α-labelings the vertices whose labels do not exceed the value λ form one of the sets of the bipartition.

- P_n always has an α-labeling
- caterpillars have an α-labeling (A. Rosa, 1966)
- path-like trees have an α-labeling (C. Barrientos, 2004)

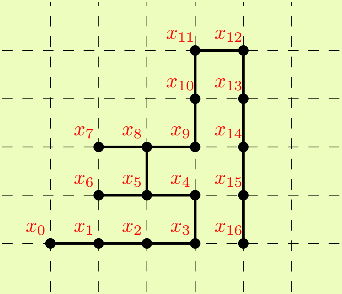
α-labeling

Figure of path-like tree.



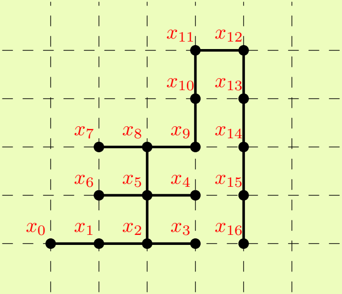
α-labeling

Figure of path-like tree.



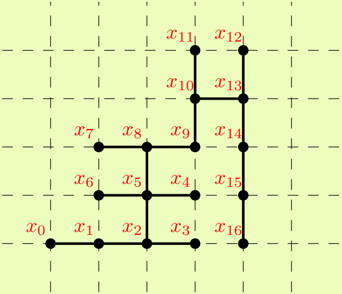
α-labeling

Figure of path-like tree.



α-labeling

Figure of path-like tree.



α-labeling $\sim$ $(a, 1)$-EAV

Lemma

Let T be a tree of order p. If T admits an α-labeling then T also admits an $(a, 1)$-EAV labeling.

• F.A. Muntaner-Batle, *Magic graphs*, Ph.D. Thesis, University Politecnica de Catalunya, 2001, Barcelona.

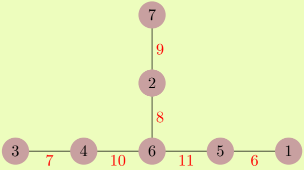
α-labeling $\sim$ $(a, 1)$-EAV

Lemma

Let T be a tree of order p. If T admits an α-labeling then T also admits an $(a, 1)$-EAV labeling.

- F.A. Muntaner-Batle, *Magic graphs*, Ph.D. Thesis, University Politecnica de Catalunya, 2001, Barcelona.

Converse of this lemma does not hold.



$(6, 1)$-EAV labeling of tree.

α-labeling $\sim$ $(3,2)$-EAV

Lemma

Let T be an α-tree. If $||A| - |B|| \leq 1$ then T is $(3,2)$-EAV.

α-labeling $\sim$ $(3,2)$-EAV

Lemma

Let T be an α-tree. If $||A| - |B|| \leq 1$ then T is $(3,2)$-EAV.

Lemma

Let T be a tree of order p. If $||A| - |B|| > 1$, then there is no $(3,2)$-EAV labeling of T.

α-labeling $\sim$ $(3,2)$-EAV

Lemma

Let T be an α-tree. If $||A| - |B|| \leq 1$ then T is $(3,2)$-EAV.

Lemma

Let T be a tree of order p. If $||A| - |B|| > 1$, then there is no $(3,2)$-EAV labeling of T.

Lemma

Let T be a tree of order p. If T does not admit α-labeling, then neither admits a $(3,2)$-EAV labeling.

α-labeling $\sim$ $(3,2)$-EAV

Theorem

A tree T is $(3,2)$-EAV if and only if T is an α-tree and $||A|-|B|| \leq 1$, where $\{A, B\}$ is the bipartition of its vertex-set.

• M. Bača and C. Barrientos, *Graceful and edge-antimagic labelings*, Ars Combin. **96** (2010), 505–513.

α-labeling $\sim$ edge-antimagic total labeling

Theorem

If (p, q) graph G has an (a, d)-EAV labeling then

(i) *G has a super $(a + p + 1, d + 1)$-EAT labeling,*

(ii) *G has a super $(a + p + q, d - 1)$-EAT labeling.*

- M. Bača, Y. Lin, M. Miller and R. Simanjuntak, *New constructions of magic and antimagic graph labelings*, Utilitas Math. **60** (2001), 229–239.

α-labeling ∼ edge-antimagic total labeling

Theorem

If (p, q) graph G has an (a, d)-EAV labeling then

(i) G *has a super* $(a + p + 1, d + 1)$*-EAT labeling,*

(ii) G *has a super* $(a + p + q, d - 1)$*-EAT labeling.*

- M. Bača, Y. Lin, M. Miller and R. Simanjuntak, *New constructions of magic and antimagic graph labelings*, Utilitas Math. **60** (2001), 229–239.

Lemma

Let T be a tree of order p. If T admits an α-labeling then T also admits an $(a, 1)$-EAV labeling.

α-labeling $\sim$ edge-antimagic total labeling

Theorem

If (p, q) graph G has an (a, d)-EAV labeling then

(i) G has a super $(a + p + 1, d + 1)$-EAT labeling,

(ii) G has a super $(a + p + q, d - 1)$-EAT labeling.

- M. Bača, Y. Lin, M. Miller and R. Simanjuntak, *New constructions of magic and antimagic graph labelings*, Utilitas Math. **60** (2001), 229–239.

Lemma

Let T be a tree of order p. If T admits an α-labeling then T also admits an $(a, 1)$-EAV labeling.

Corollary

Every α-tree also admits a super $(a, 0)$-EAT and super $(a', 2)$-EAT labeling.

α-labeling $\sim$ edge-antimagic total labeling

Theorem

If (p, q) graph G has an (a, d)-EAV labeling then

(i) *G has a super $(a + p + 1, d + 1)$-EAT labeling,*

(ii) *G has a super $(a + p + q, d - 1)$-EAT labeling.*

- M. Bača, Y. Lin, M. Miller and R. Simanjuntak, *New constructions of magic and antimagic graph labelings*, Utilitas Math. **60** (2001), 229–239.

Lemma

Let T be an α-tree. If $||A| - |B|| \leq 1$ then T is $(3, 2)$-EAV.

α-labeling $\sim$ edge-antimagic total labeling

Theorem

If (p, q) graph G has an (a, d)-EAV labeling then

(i) G has a super $(a + p + 1, d + 1)$-EAT labeling,

(ii) G has a super $(a + p + q, d - 1)$-EAT labeling.

- M. Bača, Y. Lin, M. Miller and R. Simanjuntak, *New constructions of magic and antimagic graph labelings*, Utilitas Math. **60** (2001), 229–239.

Lemma

Let T be an α-tree. If $||A| - |B|| \leq 1$ then T is $(3, 2)$-EAV.

Corollary

Every α-tree with $||A| - |B|| \leq 1$ admits a super (a, d)-EAT labeling for every $d \in \{0, 1, 2, 3\}$.

Super $(a, 3)$–EAT double star

Theorem

For the double star $S_{m,n}$, $m \neq n$ and $|m - n| \neq 1$, there is no super $(a, 3)$-EAT labeling.

- K.A. Sugeng - M. Miller - Slamin - M. Bača, *(a,d)-edge-antimagic total labelings of caterpillars*, Lecture Note Computer Science - LNCS **3330** (2005), 169–180.

Super $(a, 3)$–EAT double star

Theorem

For the double star $S_{m,n}$, $m \neq n$ and $|m - n| \neq 1$, there is no super $(a, 3)$-EAT labeling.

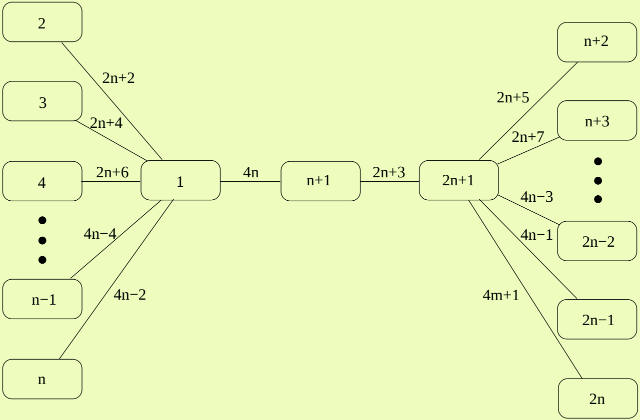
- K.A. Sugeng - M. Miller - Slamin - M. Bača, *(a,d)-edge-antimagic total labelings of caterpillars*, Lecture Note Computer Science - LNCS **3330** (2005), 169–180.

Conjecture

For the caterpillar $S_{n_1,n_2,\ldots,n_r}$, $||A| - |B|| > 1$, there is no super $(a, 3)$-EAT labeling.

Caterpillar $S_{n,2,n}$

C. Barrientos described a super $(2n+5,3)$-EAT labeling $S_{n,2,n}$, $n \geq 4$, where $||A|-|B|| = 2n-3 \geq 5$.



Construction of graceful-trees

In 1973 Stanton and Zarnke as the first developed a nontrivial algorithm for constructing graceful trees.

• R.A. Stanton and C.R. Zarnke, Labelling of balanced trees, *Proc. of the Fourth Southeastern Conference On Combinatorics, Graph Theory and Computing* (Boca Raton), 1973, pp. 479–495.

Construction of graceful-trees

Koh, Rogers and Tan graph's operation.

Let T_1 and T_2 be two graceful trees where $\{w_1, w_2, \ldots, w_m\}$ and $\{v_1, v_2, \ldots, v_n\}$ are their corresponding vertex sets.
Let $v^\star$ be an arbitrary fixed vertex in T_2.

We add m isomorphic copies of T_2 to T_1 in such a way that $v^\star$ and w_i,

$i = 1, 2, \ldots, m$ are identified.

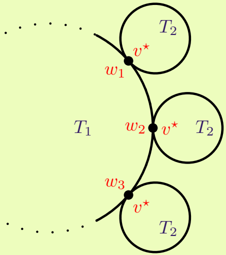
Construction of graceful-trees

Koh, Rogers and Tan graph's operation.

Let T_1 and T_2 be two graceful trees where $\{w_1, w_2, \ldots, w_m\}$ and $\{v_1, v_2, \ldots, v_n\}$ are their corresponding vertex sets.
Let $v^\star$ be an arbitrary fixed vertex in T_2.

We add m isomorphic copies of T_2 to T_1 in such a way that $v^\star$ and w_i,

$i = 1, 2, \ldots, m$ are identified.



$T_1 \triangle T_2$

Construction of graceful-trees

Theorem

If T_1 and T_2 are both graceful trees then tree $T_1 \triangle T_2$ is also graceful.

- K.M. Koh, D.G. Rogers and T. Tan, Two theorems on graceful trees, *Discrete Math.*, **25** (1979), 141–148.

Construction of α-trees

Theorem

Let T be a graceful tree of order n. If k is even positive integer then the tree $P_k \triangle T$ admits an α-labeling.

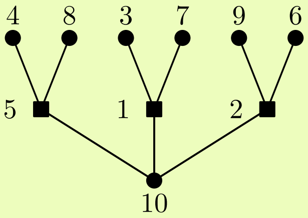
- M. Bača, A. Semaničová-Feňovčíková and M.K. Shafiq, A method to generate large classes of edge-antimagic trees, *Utilitas Math.*, in press.

Construction of α-trees

Theorem

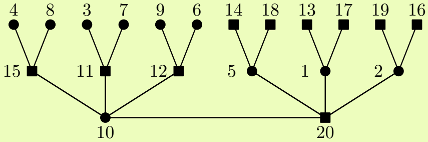
Let T be a graceful tree of order n. If k is even positive integer then the tree $P_k \triangle T$ admits an α-labeling.

- M. Bača, A. Semaničová-Feňovčíková and M.K. Shafiq, A method to generate large classes of edge-antimagic trees, *Utilitas Math.*, in press.



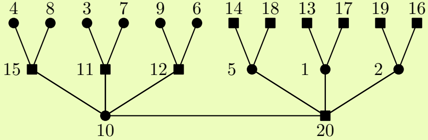
Graceful labeling of symmetrical tree ST.

Examples of α-trees

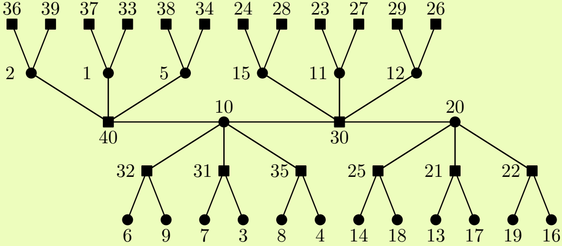


α-labeling of $P_2 \triangle ST$.

Examples of α-trees



α-labeling of $P_2 \triangle ST$.



α-labeling of $P_4 \triangle ST$.

Construction of α-trees

Theorem

Let T be a graceful tree of order n. If k is even positive integer then the tree $P_k \triangle T$ admits an α-labeling.

- M. Bača, A. Semaničová-Feňovčíková and M.K. Shafiq, A method to generate large classes of edge-antimagic trees, *Utilitas Math.*, in press.

Corollary

Let T be a graceful tree and let x be an arbitrary fixed vertex in T. If k is an even positive integer then the tree $P_k \triangle T$ admits a super (a, d)-EAT labeling for all $d \in \{0, 1, 2, 3\}$.

Construction of α-trees

Theorem

If T is an α-tree then $P_k \triangle T$ admits an α-labeling for every positive integer k.

• M. Bača, A. Semaničová-Feňovčíková and M.K. Shafiq, A method to generate large classes of edge-antimagic trees, *Utilitas Math.*, in press.

Construction of α-trees

Theorem

If T is an α-tree then $P_k \triangle T$ admits an α-labeling for every positive integer k.

- M. Bača, A. Semaničová-Feňovčíková and M.K. Shafiq, A method to generate large classes of edge-antimagic trees, *Utilitas Math.*, in press.

Corollary

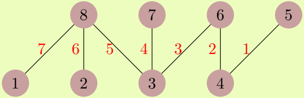
Let T be an α-tree and let $||A| - |B|| \leq 1$, $\{A, B\}$ be the bipartition of the vertex set of T. Let x be an arbitrary fixed vertex in T. Then for every positive integer k the tree $P_k \triangle T$ admits a super (a, d)-EAT labeling for all $d \in \{0, 1, 2, 3\}$.

Construction of α-trees

Theorem

Every α-tree of size q produces a super (a, d)-EAT tree of size $2q$, for every $d \in \{0, 1, 2, 3\}$.

- M. Bača and C. Barrientos, *Graceful and edge-antimagic labelings*, Ars Combin. **96** (2010), 505–513.



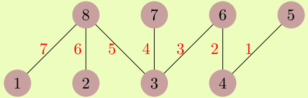
α-labeling of caterpillar.

Construction of α-trees

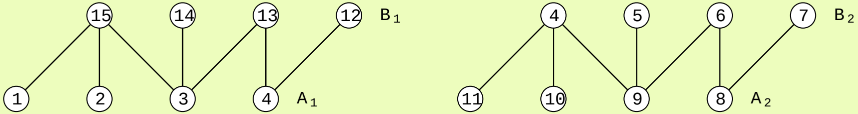
Theorem

Every α-tree of size q produces a super (a, d)-EAT tree of size $2q$, for every $d \in \{0, 1, 2, 3\}$.

- M. Bača and C. Barrientos, *Graceful and edge-antimagic labelings*, Ars Combin. **96** (2010), 505–513.



α-labeling of caterpillar.



Certain classes of graceful trees

Caterpillars

- A. Rosa, *On certain valuations of the vertices of a graph*, Internat. Symposium, Rome, 1966.

Certain classes of graceful trees

Caterpillars

• A. Rosa, *On certain valuations of the vertices of a graph*, Internat. Symposium, Rome, 1966.

Lobsters

• J.G. Wang, D.J. Jin, X.G. Lu and D. Zhang, The gracefulness of a class of lobster trees, *Mathematical and Computer Modelling* **20** (1994), 105–110.
• W.C. Chen, H.I. Lu and Y.N. Yeh, Operation of interlaced trees and graceful trees, *Southeast Asian Bulletin of Math.* **4** (1997), 337–348.
• D. Mishra and P. Panigrahi, Graceful lobsters obtained by component moving of diameter four trees, *Ars Combin.* **81** (2006), 129–147.
• D. Mishra and P. Panigrahi, Some graceful lobsters with both odd and even degree vertices on the central path, *Utilitas Math.* **74** (2007), 155–177.

Certain classes of graceful trees

Symmetrical trees

• J.C. Bermond and D. Sotteau, Graph decompositions and G-design, Proc. 5th British Combin. Conf., 1975, *Congr. Numer.* **15** (1976), 53–72.

• Poljak and M. Sûra, An algorithm for graceful labeling of a class of symmetrical trees, *Ars Combin.* **14** (1982), 57–66.

Certain classes of graceful trees

Symmetrical trees

• J.C. Bermond and D. Sotteau, Graph decompositions and G-design, Proc. 5th British Combin. Conf., 1975, *Congr. Numer.* **15** (1976), 53–72.

• Poljak and M. Sûra, An algorithm for graceful labeling of a class of symmetrical trees, *Ars Combin.* **14** (1982), 57–66.

Banana trees

• V. Bhat-Nayak and U. Deshmukh, New families of graceful banana trees, *Proc. Indian Acad. Math. Sci.* **106** (1996), 201–216.
• M. Murugan and G. Arumugan, Are banana trees graceful?, *Math. Ed. (Siwan)* **35** (2001), 18–20.

Certain classes of super EAT trees

By $\mathcal{F}$ we denote the family of graceful trees that contains caterpillars, symmetrical trees, special classes of lobsters and special classes of banana trees.

Corollary

Let $T \in \mathcal{F}$ and x be an arbitrary fixed vertex in T.

(i) *For an even positive integer k the tree $T_1 = P_k \triangle T$ admits a super (a, d)-EAT labeling for all $d \in \{0, 1, 2, 3\}$.*

(ii) *For every positive integer n the tree $T_{n+1} = P_{k_n} \triangle T_n$, where k_n is a positive integer and x_n is an arbitrary fixed vertex in T_n, admits a super (a, d)-EAT labeling for all $d \in \{0, 1, 2, 3\}$.*

Thank you for your attention