Deciding first-order properties for sparse graphs

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Grafy 2010

Meta-algorithms

Algorithms for wide classes of problems.

Example (Courcelle)

Any property expressible in Monadic Second-Order Logic can be tested in linear time for graphs with bounded treewidth. Includes

- perfect matching
- chromatic number
- hamiltonicity
- . . .

Other sparse graph classes?

Deciding 3-colorability of planar graphs is NP-complete

$$\Longrightarrow$$

Courcelle's result is unlikely to hold for planar graphs.

$$\Longrightarrow$$

Smaller class of problems is needed.

First-order properties

Properties expressible by formulas with quantification over *vertices* (not their sets).

Example

G contains a triangle:
$(\exists x, y, z)\ x \neq y \wedge x \neq z \wedge y \neq z \wedge E(x, y) \wedge E(x, z) \wedge E(y, z)$

Example

G contains a dominating set of size at most two:
$(\exists x, y)(\forall z)\ z = x \vee z = y \vee E(x, z) \vee E(y, z)$

Algorithms

- Property defined by a formula with k quantifiers can be decided in $O(n^k)$.
- Finding subgraphs on k vertices: $O(n^{\omega k/3})$ (Nešetřil, Poljak)
- FPT (complexity $f(k)n^{O(1)}$) considered unlikely in general.

FPT for sparse graphs

But, subgraphs can be found in linear time in planar graphs (Eppstein). More generally, FPT algorithms exist for

- FO properties in graphs with locally bounded treewidth (Frick and Grohe)
- FO properties in graphs excluding (locally) a minor (Dawar, Grohe and Kreutzer)
- finding subgraphs in graphs with bounded expansion and nowhere-dense graphs (Nešetřil and Ossona de Mendez)

Bounded expansion, nowhere dense

“Structurally sparse” graphs. Many equivalent definitions, e.g.,

Definition

Hereditary class $\mathcal{C}$ of graphs has bounded expansion if there exist constants $c_0, c_1, c_2, \ldots$ satisfying the following:
For any graph H and an integer $k \geq 0$, if the graph obtained from H by subdividing each edge by exactly k vertices belongs to $\mathcal{C}$, then $\delta(H) \leq c_k$.

Includes:

- graphs with bounded maximum degree
- planar graphs, graphs excluding a fixed minor
- graphs excluding a fixed topological minor
- graphs drawn with bounded # of crossings on each edge
- ...

Bounded expansion, nowhere dense

"Structurally sparse" graphs. Many equivalent definitions, e.g.,

Definition

Hereditary class $\mathcal{C}$ of graphs is nowhere dense if there exist constants $c_0, c_1, c_2, \ldots$ satisfying the following:
For an integer $n > 0$ and an integer $k \geq 0$, if the graph obtained from K_n by subdividing each edge by exactly k vertices belongs to $\mathcal{C}$, then $n \leq c_k$.

Includes:

- classes with bounded expansion
- graphs excluding locally a fixed minor/topological minor
- locally bounded expansion
- ...

Our results

- A linear-time algorithm for testing FO properties on graphs with bounded expansion.
- A semidynamic data structure for this problem:
  - initialized with a graph G with bounded expansion, in linear time
  - answer FO queries (with a bounded # of quantifiers) in $O(1)$
  - add or remove edges in $O(1)$, as long as
  - the represented graph is a subgraph of G
- A semidynamic data structure for finding subgraphs in nowhere-dense graphs.
- A fully dynamic data structure for finding bounded length paths in nowhere-dense graphs (polylogarithmic update time).

Low treedepth colorings

Definition

A vertex coloring is k-low-treedepth if the union of any $t \leq k$ color classes induces a graph of treedepth at most t.

Definition

A graph has *treedepth* at most t if it is a subgraph of a closure of a rooted forest of depth t.

Low treedepth colorings

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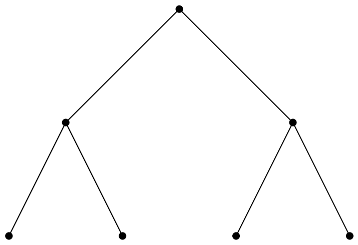

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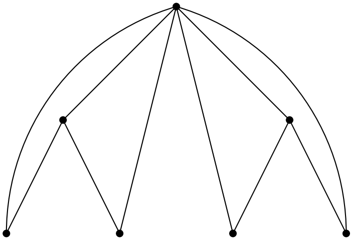

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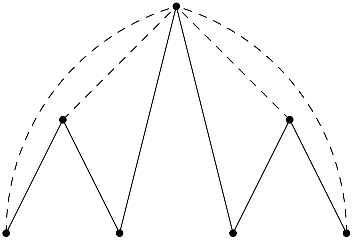

Low treedepth colorings

Theorem

Let $\mathcal{C}$ be a class of graphs with bounded expansion. Then, for every k there exists c such that every graph $G \in \mathcal{C}$ has a k-low-treedepth coloring by at most c colors. This coloring can be found in $O(|V(G)|)$ time.

Subgraphs in linear time

Testing whether G contains a subgraph H, with $|V(H)| = k$ (Nešetřil and Ossona de Mendez):

- find a k-low-treedepth coloring of G by at most c colors
- for all $\binom{c}{k}$ choices of colors of $V(H)$, test whether H appears in the subgraph of G induced by these colors.

$$\text{treewidth} \leq \text{treedepth} \implies \text{linear time}$$

Quantifier elimination

Theorem

Let $\mathcal{C}$ be a class of graphs with bounded expansion and $\varphi(x_1, x_2, \ldots, x_n) = (\exists x_0)\ \psi(x_0, x_1, \ldots, x_n)$ a first-order formula, where ψ is quantifier-free. There exists a class of graphs $\mathcal{C}'$ with bounded expansion and a quantifier-free formula $\varphi'(x_1, \ldots, x_n)$ with the following property:
For every graph $G \in \mathcal{C}$, we can in find (in linear time) a graph $G' \in \mathcal{C}'$ with $V(G') = V(G)$ such that for any $v_1, \ldots, v_n \in V(G)$,

$$G \models \varphi(v_1, v_2, \ldots, v_n) \text{ if and only if } G' \models \varphi'(v_1, v_2, \ldots, v_n).$$

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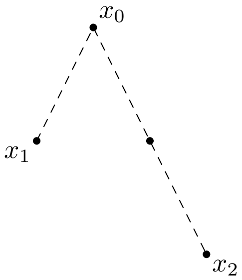
graph = directed multigraph with colors on vertices and edges

Quantifier elimination – example

$$(\exists x_0) E(x_0, x_1) \wedge E(x_0, x_2)$$



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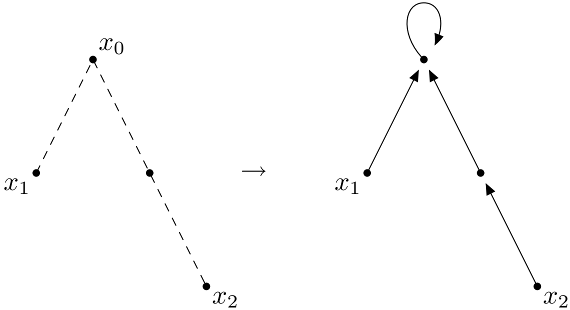

$$f(x_1) = f(f(x_2)) \wedge x_1 \neq f(x_2) \quad \wedge \quad x_1 \neq f(x_1) \wedge f(x_1) = f(f(x_1)) \wedge$$
$$E(x_1, f(x_1)) \quad \wedge \quad E(x_2, f(x_1))$$

Application

Linear-time algorithm for 5-coloring graphs on a fixed surface:

- a finite number of obstructions (Thomassen; Postle and Thomas)
- algorithm:
  - find a reducible configuration and reduce it
  - if no obstruction is created, continue
  - otherwise, split the original graph along the subgraph corresponding to the obstruction
- necessary to test whether an obstruction is created, in constant time

Open problem

Problem

Find an FPT for FO properties in nowhere-dense graph classes!