A generalization of Kyš's and Plesník's construction of goal-minimally k-diametric graphs

Štefan Gyürki[†] Beáta Štupáková[‡]

[†]FCFT, STU in Bratislava, Slovakia

[‡]FMPI, Comenius University in Bratislava, Slovakia

Lednice, 2010

Definition

Definition

A graph G is said to be **k-GMD**, if the diameter of G is equal to k, and for every edge $uv \in E(G)$

$$d_{G-uv}(x, y) > k \iff \{u, v\} = \{x, y\}.$$

Examples

Example

The complete graph K_n is 1-GMD for any $n \geq 3$.

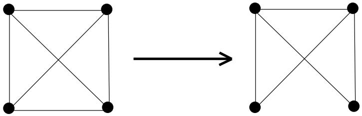

Figure: The graph K_4 before and after removing an edge.

Basic properties

Theorem (Kyš, Gliviak, Plesník)

Let G be a k-GMD graph of order at least 3. Then

Basic properties

Theorem (Kyš, Gliviak, Plesník)

Let G be a k-GMD graph of order at least 3. Then

1. *the girth of G is $k+2$,*

Basic properties

Theorem (Kyš, Gliviak, Plesník)

Let G be a k-GMD graph of order at least 3. Then

1. *the girth of G is $k + 2$,*
2. *every edge of G lies in a cycle of length $k + 2$,*

Basic properties

Theorem (Kyš, Gliviak, Plesník)

Let G be a k-GMD graph of order at least 3. Then

1. *the girth of G is $k+2$,*
2. *every edge of G lies in a cycle of length $k+2$,*
3. *for any two non-adjacent vertices u and v there are at least two internally disjoint $u-v$ paths of length not exceeding k.*

Basic properties

Theorem (Kyš, Gliviak, Plesník)

Let G be a k-GMD graph of order at least 3. Then

1. *the girth of G is $k+2$,*
2. *every edge of G lies in a cycle of length $k+2$,*
3. *for any two non-adjacent vertices u and v there are at least two internally disjoint $u-v$ paths of length not exceeding k.*

Conversely, if 1 and 3 hold for a graph G with diameter k, then G is k-GMD.

Kyš's conjecture

Conjecture [Kyš, 1980]

For every positive integer k there exists a k-GMD graph.

Known results

Table: The situation before 2010.

Author (Year)	Values of k	
	Sporadic examples	Infinite families
Kyš (1980)	1,2,3,4,6	1,2,4
Plesník (2006)	5,7,8,10,12,14	6
Š. Gy. (2008)	9,13	5
Š. Gy. (2009)	16, 18–24, 26	

Connecting the leaves of two trees

Known infinite classes with even diameter

Connecting the leaves of two trees

Known infinite classes with even diameter

- 2-GMD graphs

Connecting the leaves of two trees

Known infinite classes with even diameter

- 2-GMD graphs
- 4-GMD graphs [Kyš]

Connecting the leaves of two trees

Known infinite classes with even diameter

- 2-GMD graphs
- 4-GMD graphs [Kyš]
- 6-GMD graphs [Plesník]

Connecting the leaves of two trees

Known infinite classes with even diameter

- 2-GMD graphs
- 4-GMD graphs [Kyš]
- 6-GMD graphs [Plesník]

Plesník also constructed some k-GMD graphs by connecting two binary trees for $k = 8, 10, 12$ and 14.

1. *k*-GMD graphs
   - Definition and examples
   - Basic properties and known results

2. A construction
   - Known classes
   - Diameter 6
   - Diameter 8

Diameter 6

Let T_1, T_2 be two trees of height 2 such that every vertex has q childs, where $q = p^r$ for some p prime.

Diameter 6

Let T_1, T_2 be two trees of height 2 such that every vertex has q childs, where $q = p^r$ for some p prime.

Leaves u_{ab} and $v_{\alpha\beta}$ are joined by a path of length 2 if and only if $\beta = a \cdot \alpha + b$ holds in $GF(q)$.

Diameter 6

Let T_1, T_2 be two trees of height 2 such that every vertex has q childs, where $q = p^r$ for some p prime.

Leaves u_{ab} and $v_{\alpha\beta}$ are joined by a path of length 2 if and only if $\beta = a \cdot \alpha + b$ holds in $GF(q)$.

Theorem

The resulting graph is 6-GMD for every q, where q is a power of a prime.

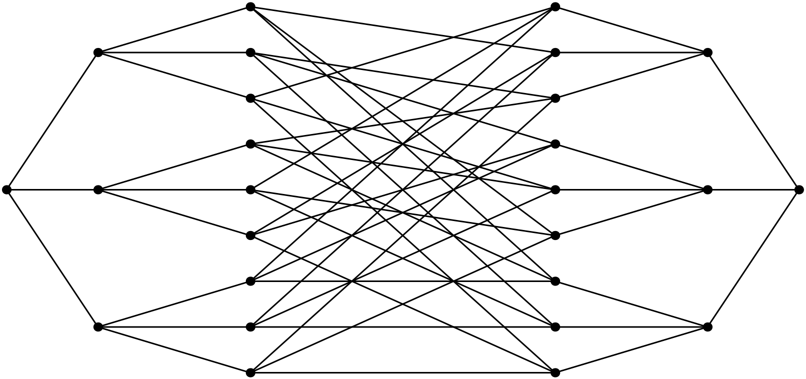

Figure: A 6-GMD graph.

Diameter 8

Let T_1, T_2 be two trees of height 3 such that the root has $q = p^2$ childs, and the other vertices have p childs.

Diameter 8

Let T_1, T_2 be two trees of height 3 such that the root has $q = p^2$ childs, and the other vertices have p childs.

Leaves u_{ab} and $v_{\alpha\beta}$ are joined by a path of length 2 if and only if $\beta = (a \cdot \alpha + b) \cdot \bar{x}$ holds in $GF(q) \cong \mathbb{Z}_p[x]/(P_2(x))$, where $P_2(x)$ is a polynomial of degree 2 irreducible over $\mathbb{Z}_p[x]$ and $\bar{x}$ its root.

Diameter 8

Let T_1, T_2 be two trees of height 3 such that the root has $q = p^2$ childs, and the other vertices have p childs.

Leaves u_{ab} and $v_{\alpha\beta}$ are joined by a path of length 2 if and only if $\beta = (a \cdot \alpha + b) \cdot \bar{x}$ holds in $GF(q) \cong \mathbb{Z}_p[x]/(P_2(x))$, where $P_2(x)$ is a polynomial of degree 2 irreducible over $\mathbb{Z}_p[x]$ and $\bar{x}$ its root.

Theorem

The resulting graph is 8-GMD for every $q = p^2$.

Known results

Table: The present situation.

Author (Year)	Values of k	
	Sporadic examples	Infinite families
Kyš (1980)	1,2,3,4,6	1,2,4
Plesník (2006)	5,7,8,10,12,14	6
Š. Gy. (2008,09)	9,13,16,18–24,26	5
Š. Gy., B.Š. (2010)		8, 10

Thank You For Your Attention.