Generalized fractional total coloring of complete graphs

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- A *graph property* $\mathcal{P}$ is a non-empty isomorphism-closed subclass of $\mathcal{I}$
- A property P is called *additive* iff $G \cup H \in \mathcal{P}$ whenever $G \in \mathcal{P}$ and $H \in \mathcal{P}$
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We use the following standard notations for specific additive and hereditary properties:

- $\mathcal{O} = \{G \in \mathcal{I} : E(G) = \emptyset\}$
- $\mathcal{O}^k = \{G \in \mathcal{I} : \chi(G) \leq k\}$
- $\mathcal{D}_k = \{G \in \mathcal{I} :$ each subgraph of G contains a vertex of degree at most $k\}$
- $\mathcal{I}_k = \{G \in \mathcal{I} :$ G does not contain $K_{k+2}\}$
- $\mathcal{O}_k = \{G \in \mathcal{I} :$ each component of G has at most $k + 1$ vertices$\}$
- $\mathcal{S}_k = \{G \in \mathcal{I} : \Delta(G) \leq k\}$

where $\chi(G)$ is the *chromatic number* and $\Delta(G)$ the *maximum degree* of the graph $G = (V, E)$.

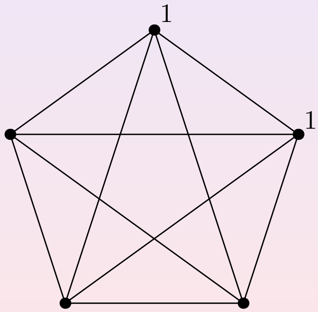
In the following let $\mathcal{P}, \mathcal{Q}$ be two additive and hereditary properties.

Definition 1

Let $G = (V, E)$ be a graph. Let $r, s \in \mathbb{N}$ and $2s \leq r$. A $\frac{r}{s}$*-fractional* $(\mathcal{P},\mathcal{Q})$*-total coloring* of G is a mapping $f : V \cup E \rightarrow \binom{\{0,1,\dots,r-1\}}{s}$ such that for each color i ($i \in \{0, 1, \dots, r-1\}$) all vertices of color i induce a subgraph from the property $\mathcal{P}$, all edges of color i induce a subgraph from the property $\mathcal{Q}$, moreover the vertices and incident edges have assigned disjoint sets of colors. The *fractional* $(\mathcal{P},\mathcal{Q})$*-total chromatic number* is $\chi''_{f,\mathcal{P},\mathcal{Q}}(G) = inf\{\frac{r}{s}$ such that there exists $\frac{r}{s}$ - fractional $(\mathcal{P}, \mathcal{Q})$- total coloring of $G\}$.

Example:

$$\chi''_{f,\mathcal{O}_1,\mathcal{I}_1}(K_5) \le \frac{7}{2}$$



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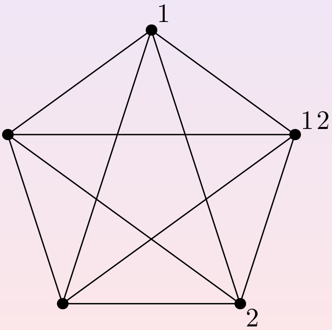

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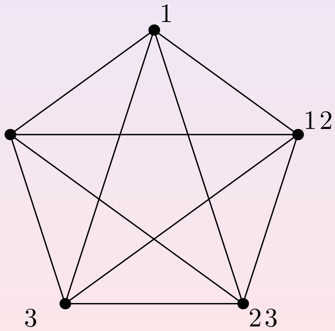

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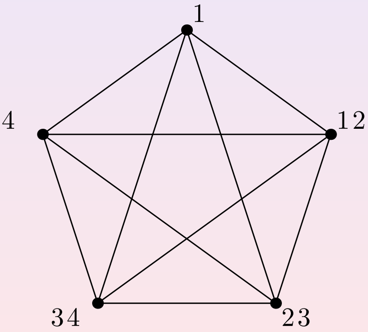

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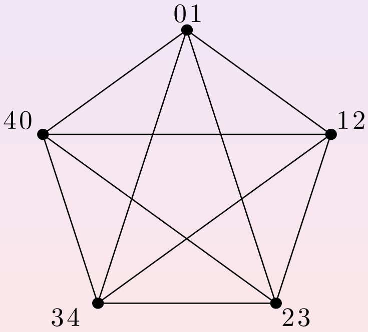

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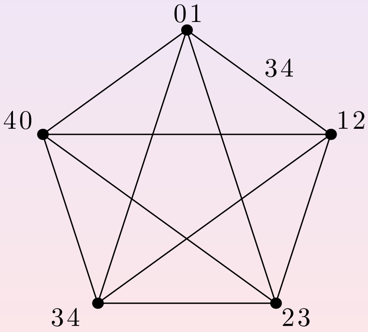

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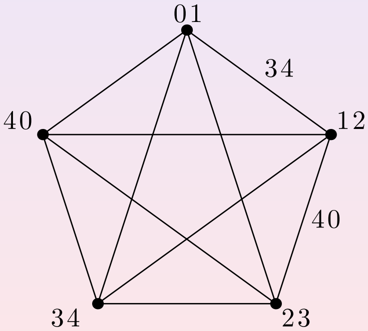

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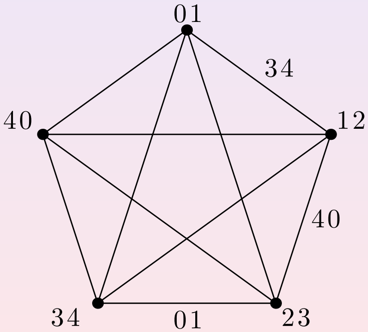

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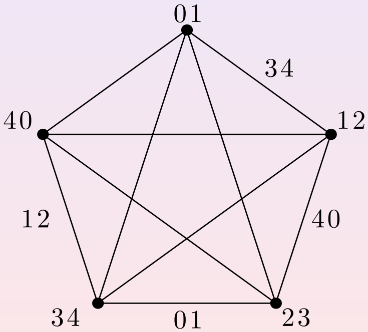

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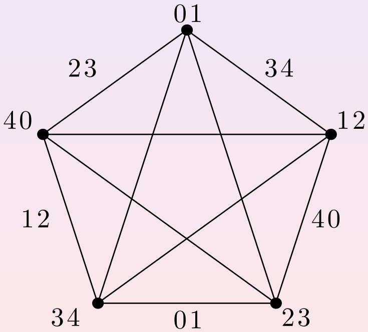

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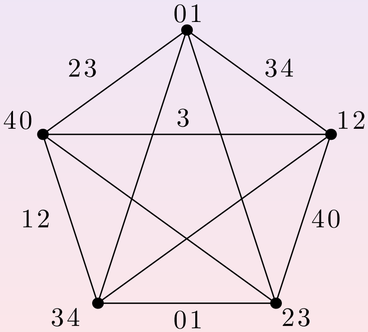

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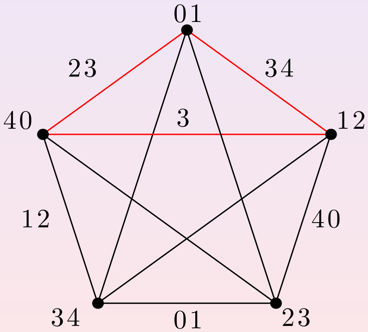

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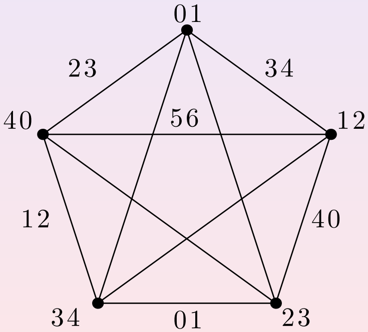

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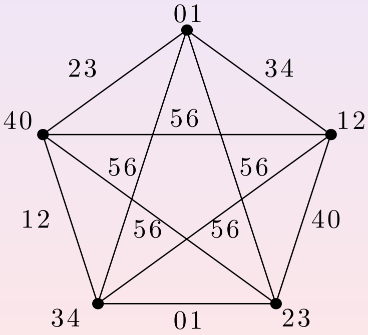

($\mathcal{P}$,$\mathcal{Q}$)-independent set is a subset of $V \cup E$ such that the vertices in this set induce graph from the property $\mathcal{P}$, edges induce graph from the property $\mathcal{Q}$ and moreover vertices and edges are not incident.

Definition 2

Let $I_1, I_2, \ldots, I_t, \quad t \in \mathbb{N}$ be all (maximal) ($\mathcal{P}$,$\mathcal{Q}$)-independent sets in G. A *fractional ($\mathcal{P}$,$\mathcal{Q}$)-total coloring* of G is a mapping g, which assigns to each set $I_j, \quad j = 1, \ldots, t$ a non-negative weight $g(I_j)$ such that $\sum\limits_{u \in I_j} g(I_j) \geq 1$ for each element $u \in V \cup E$. The *fractional ($\mathcal{P}$,$\mathcal{Q}$)-total chromatic number* $\chi''_{f,\mathcal{P},\mathcal{Q}}(G)$ of G is the least total weight of the fractional ($\mathcal{P}$,$\mathcal{Q}$)-total coloring of G.

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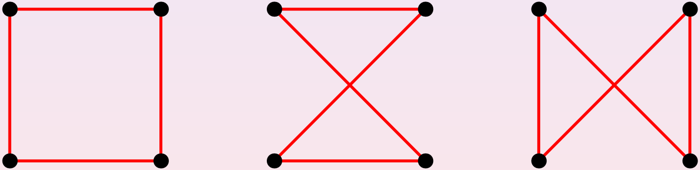
Maximal independent sets of K_4, where $\mathcal{P} = \mathcal{O}_1, \mathcal{Q} = \mathcal{I}_1$ with no red vertex:



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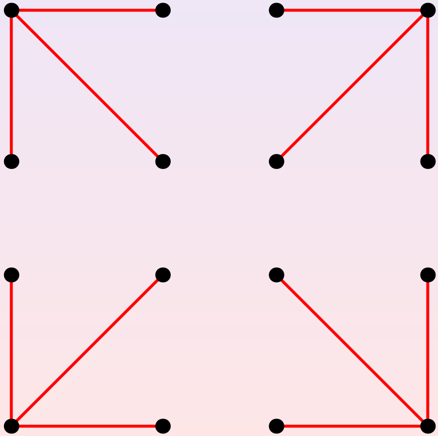

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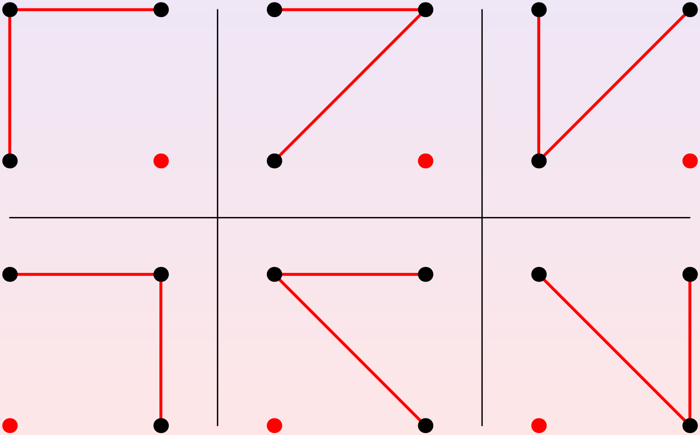

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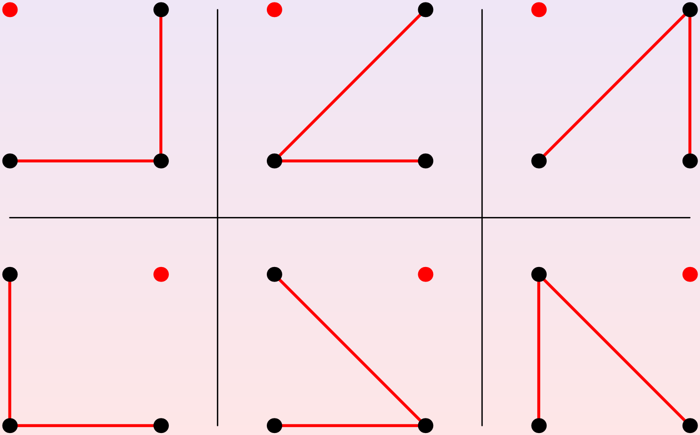

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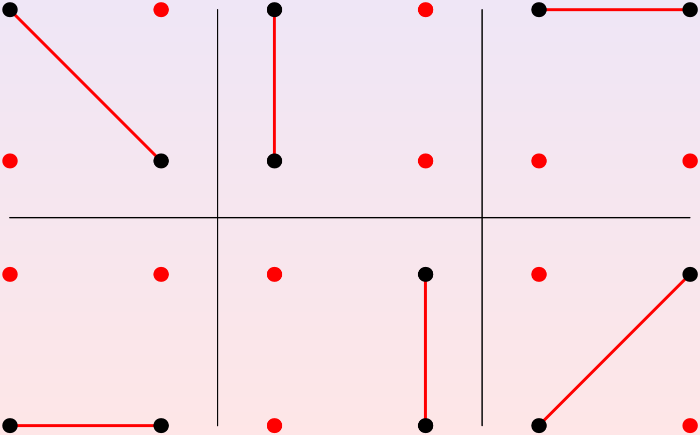

The fractional $(\mathcal{P},\mathcal{Q})$-total chromatic number $\chi''_{f,\mathcal{P},\mathcal{Q}}(G)$ can be obtained as a solution of the following problem of linear programming:

$$g : \sum_{j=1}^{t} g(I_j) \rightarrow \min$$

$$(1) \quad \sum_{u \in I_j} g(I_j) \geq 1, \ \ \forall u \in V \cup E$$

$$g(I_j) \geq 0, \ \ \forall j = 1, \ldots, t$$

- Now we consider some $\frac{r}{s}$-fractional $(\mathcal{P},\mathcal{Q})$-total coloring of K_n
- $k = c(\mathcal{P}) = sup\{i : K_{i+1} \in \mathcal{P}\}$
- $l = c(\mathcal{Q}) = sup\{i : K_{i+1} \in \mathcal{Q}\}$
- x_i - a number of colors used exactly for i vertices, $i = 0, 1, \ldots, k+1$
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Let $x'_i := \frac{x_i}{s}$. The main result:

$$(1) \quad \Leftrightarrow \quad (2)$$

$$g : \sum_{j=1}^{t} g(I_j) \to \min \qquad h : \sum_{i=0}^{k+1} x'_i \to \min$$

$$\sum_{u \in I_j} g(I_j) \geq 1, \quad \forall u \in V \cup E \qquad \sum_{i=0}^{k+1} i x'_i \geq n$$

$$g(I_j) \geq 0, \quad \forall j = 1, \dots, t \qquad \sum_{i=0}^{k+1} a_i x'_i \geq \frac{n(n-1)}{2}$$

$$x'_i \geq 0, \quad \forall i = 0, \dots, k+1$$

Theorem

Let $\mathcal{P}$, $\mathcal{Q}$ be two additive and hereditary properties such that $c(\mathcal{Q}) \geq c(\mathcal{P}) + 2$. Then there exists n_0 such that for each $n \geq n_0$: $\chi''_{f,\mathcal{P},\mathcal{Q}}(K_n) = \frac{n}{k+1}$

Corollary

Let G be an arbitrary graph and $\mathcal{P}$, $\mathcal{Q}$ be two additive and hereditary properties such that $c(\mathcal{Q}) \geq c(\mathcal{P}) + 2$. Then there exists n_0 such that for each $n \geq n_0$: $\chi''_{f,\mathcal{P},\mathcal{Q}}(G) \leq \frac{n}{k+1}$

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Example:

$$\chi''_{f,\mathcal{O}_1,\mathcal{I}_1}(K_n) = \begin{cases} 3, & n = 3, 4, \\ \frac{10}{3}, & n = 5, 6, \\ \frac{n}{2}, & n \geq 7. \end{cases}$$

Thank you for your attention:-)