Perfect matchings in cubic graphs

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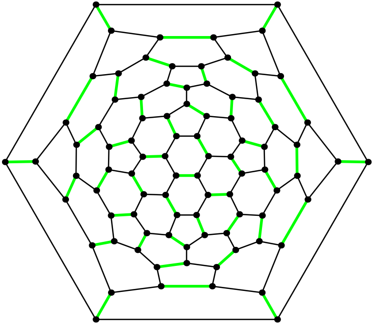
What is it?



Conjecture (Lovász & Plummer 70's)

There exists a constant $c > 0$, such that any n-vertex cubic bridgeless graph contains at least 2^{cn} perfect matchings.

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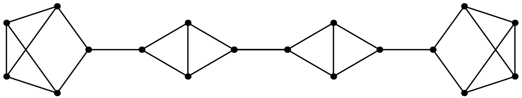

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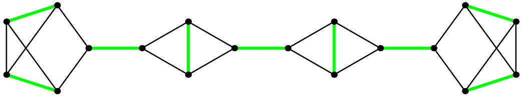

Let G be a bipartite cubic bridgeless graph with n vertices.

- $m(G) \geq n/2$ (Sinkhorn 1969)
- $m(G) \geq n/2 + 2$ (Minc 1969)
- $m(G) \geq 3n/2 - 3$ (Hartfiel & Crosby 1971)
- $m(G) \geq 6 \cdot (4/3)^{n/2-3}$ (Voorhoeve 1979)

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Theorem (Schrijver 1998)

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Theorem (Chudnovsky & Seymour 2008)

Every planar cubic bridgeless graph with n vertices contains at least $2^{n/655978752}$ perfect matchings.

Theorem (Edmonds, Lovász & Pulleyblank; Naddef 1982)

Every cubic bridgeless graph with n vertices contains at least $n/4+2$ perfect matchings.

Theorem (Kráľ, Sereni & Stiebitz 2008)

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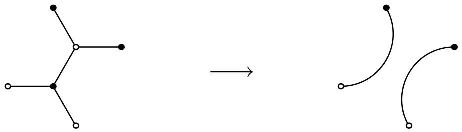
Theorem (Esperet, K. & Kráľ 2009)

For any $a > 0$ there exists a constant b such that every cubic bridgeless graph with n vertices contains at least $an - b$ perfect matchings.

Splitting an edge



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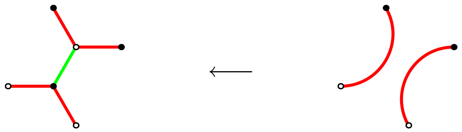

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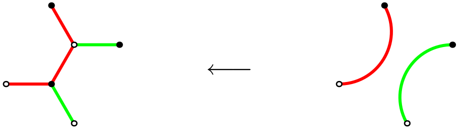

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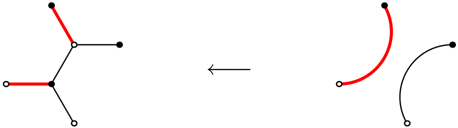

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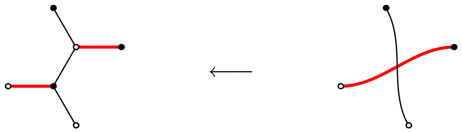

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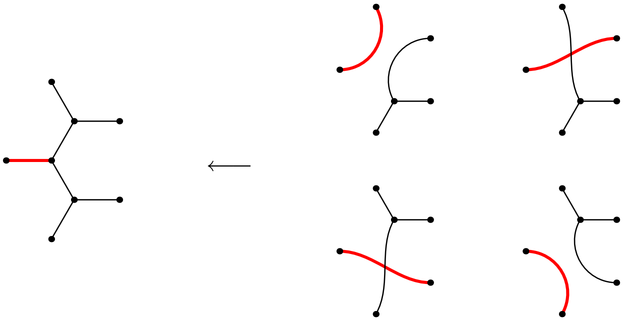

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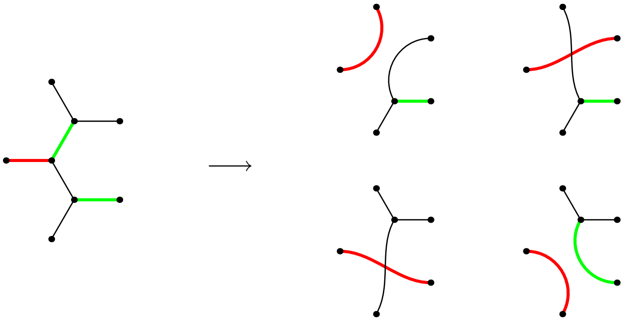

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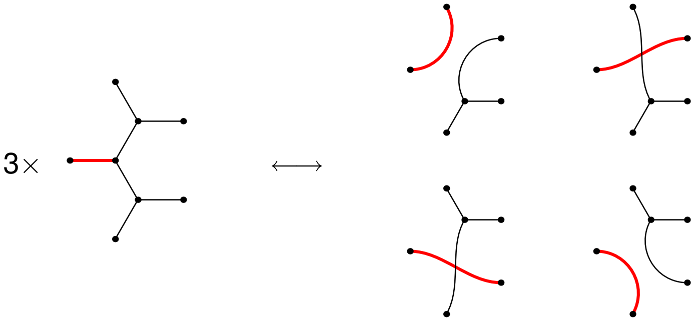


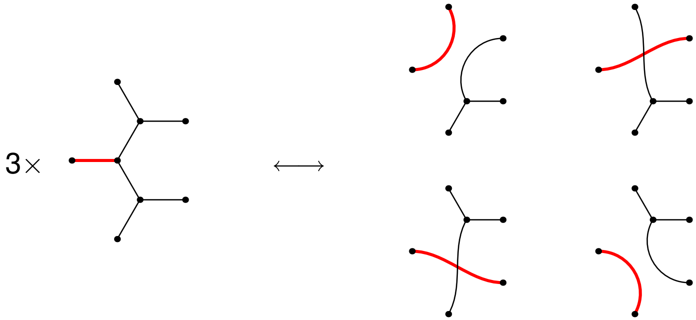
Splitting an edge

$$m_e(n) \geq \frac{4}{3} \cdot m_e(n-2).$$

Theorem (Voorhoeve 1979)

Let G be a bipartite cubic graph with n vertices. Then $m(G) \geq 6 \cdot (4/3)^{n/2-3}$.

Troubles in the general case



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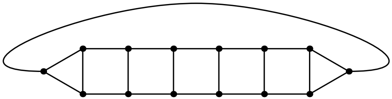

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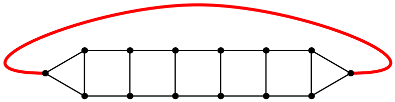

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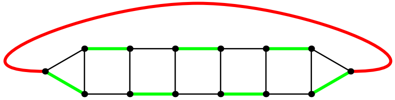
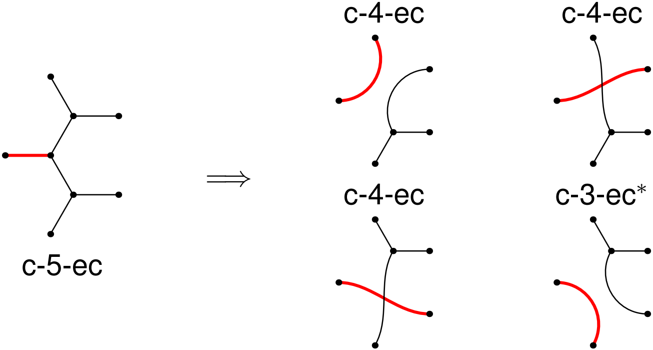


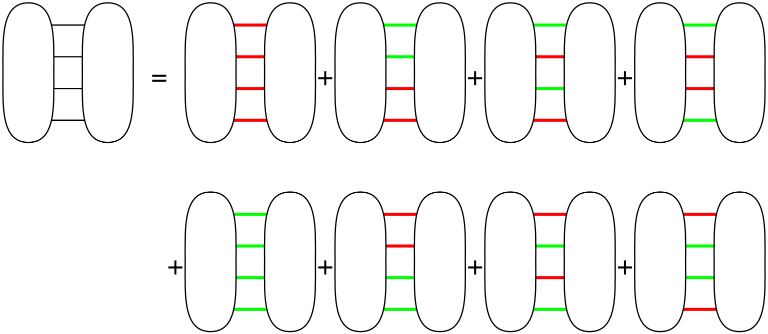


c-k-ec: a cyclically k-edge-connected graph
c-k-ec*: a cyclically k-edge-connected graph, with no cyclic k-edge cut containing the forbidden edge e

Lemma

Let G be an n-vertex cyclically 3-edge-connected cubic graph and e an edge of G that is not contained in any cyclic 3-edge-cut of G. The number of perfect matchings of G that avoids e is at least $n/8$.

Graphs with cyclic 4-edge-cuts



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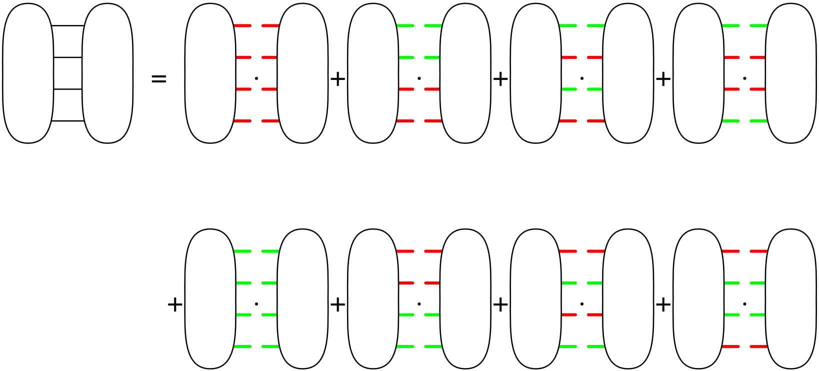
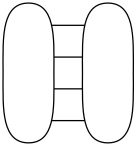




$$= \sum_{i=1}^{8} x_i y_i$$

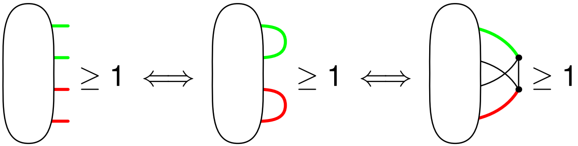
If $x_i \geq 1$ and $y_i \geq 1$ for some $i \in I \subseteq \{1, 2, \ldots, 8\}$, then

$$x_i y_i \geq x_i + y_i - 1$$

and thus

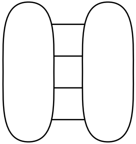
$$\sum_{i \in I} x_i y_i \geq \sum_{i \in I} x_i + \sum_{i \in I} y_i - 8.$$

Graphs with cyclic 4-edge-cuts



Lemma (Esperet, Kráľ, Škoda, Škrekovski 2008)

Let G be a cyclically 4-edge-connected cubic graph and e and f two edges of G. G contains no perfect matchings avoiding e and containing f if and only if the graph $G \setminus \{e, f\}$ is bipartite and the end-vertices of e are in one color class while and the end-vertices of f are in the other.

 $= \sum_{i=1}^{8} x_i y_i$

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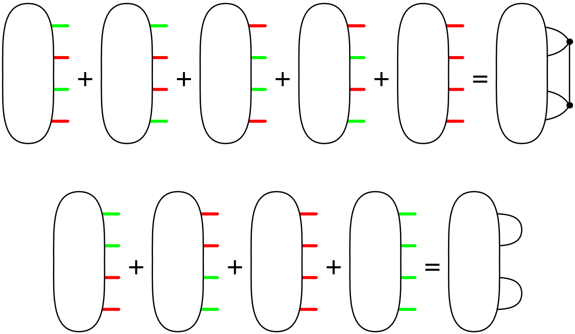
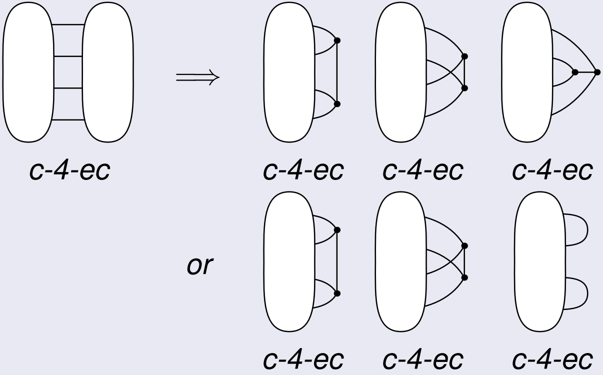


Lemma



Thank you for your attention!