On the crossing numbers of join products

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Graph $G = (V, E)$ – simple graph

A drawing $D(G)$ of the graph G is a representation of G in the plane such that its vertices are represented by distinct points and its edges by simple continuous arcs connecting the corresponding point pairs.

Crossing number $cr(G)$ of a graph G is the minimum number of edge crossings in any drawing of G in the plane (where three edges do not cross in a point).

Optimal drawing – good drawing

– no edge crosses itself
– no two edges cross more than once
– no two edges incident with the same vertex cross

$cr(G)$ – crossing number of G $D(G)$ – drawing of G (usally D)

$cr_D(G)$ – number of crossings in the drawing D of the graph G

clearly:

$$cr(G) \leq cr_D(G)$$
$$H \subset G \;\Rightarrow\; cr(H) \leq cr(G)$$

Let G_i and G_j be two edge disjoint subgraphs of G. We use the next notation:

$cr_D(G_i, G_j)$ is the number of crossings between edges of G_i and G_j.

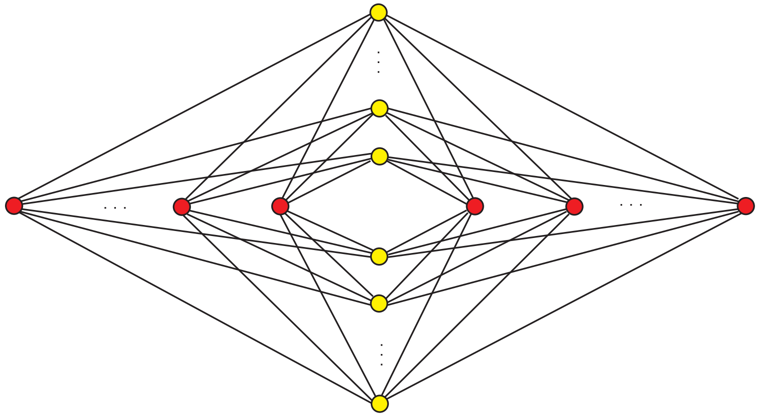
It is easy to see that

$$cr_D(G_i \cup G_j) = cr_D(G_i) + cr_D(G_j) + cr_D(G_i, G_j)\,.$$

$cr(K_n) \leq \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor$ $cr(K_5) = 1$

Equality: $n = 5, 6, \ldots, 10, 11, 12$ $cr(K_{11}) = 100, cr(K_{12}) = 150$
(Richter, Pan, 2007)

K. Zarankiewicz, 1954 $cr(K_{m,n}) = \left\lfloor\frac{m}{2}\right\rfloor\left\lfloor\frac{m-1}{2}\right\rfloor\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor$



In fact: $cr(K_{m,n}) \leq \left\lfloor\frac{m}{2}\right\rfloor\left\lfloor\frac{m-1}{2}\right\rfloor\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor$

D. Kleitman, 1970
$cr(K_{m,n}) = Z(m,n) = \left\lfloor\frac{m}{2}\right\rfloor\left\lfloor\frac{m-1}{2}\right\rfloor\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor$ if $\min\{m,n\} \leq 6$

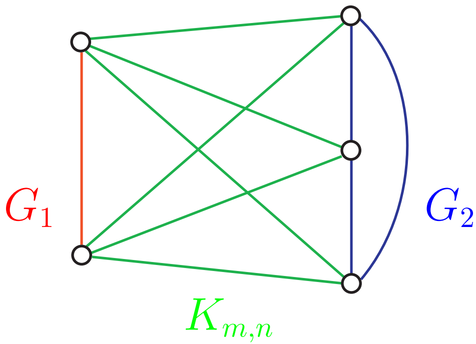
The join product of two graphs G_1 and G_2, denoted by

$$G_1 + G_2\,,$$

is obtained from vertex-disjoint copies of G_1 and G_2 by adding all edges between $V(G_1)$ and $V(G_2)$.

For $|V(G_1)| = m$ and $|V(G_2)| = n$, the edge set of $G_1 + G_2$ is the union of disjoint edge sets of the graphs G_1, G_2, and the complete bipartite graph $K_{m,n}$.

$G_1 + G_2$



Kulli, Muddebihal (2001)
The characterisation of all pairs of graphs which join is planar graph.

P_n – path on n vertices
C_n – cycle on n vertices

M. K. (2007):

$P_n + P_m$, $P_n + C_m$, and $C_n + C_m$.

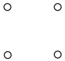
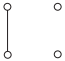
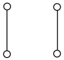
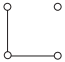
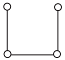
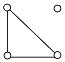
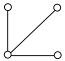
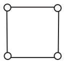
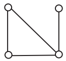
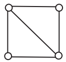
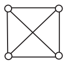
$$cr(P_m + P_n) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor, \qquad \min\{m,n\} \leq 6$$

$$cr(P_m + C_n) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 1, \quad \min\{m,n\} \leq 6$$

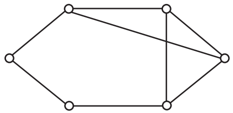
$$cr(C_m + C_n) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2, \quad \min\{m,n\} \leq 6$$

The crossing numbers of $G + P_n$ and $G + C_n$ for all graphs G of order at most three and for some graphs of order four.

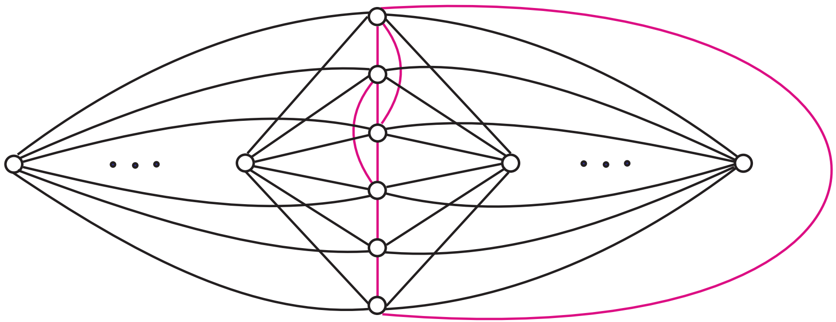
Š. Schrötter, M. K. (2009, 2010):

G_i		$cr(G_i + nK_1)$	$cr(G_i + P_n)$	$cr(G_i + C_n)$
G_1		$Z(4,n)$	$Z(4,n)$	$Z(4,n)$
G_2		$Z(n,4)$	$Z(4,n)$	$Z(4,n)$
G_3		$Z(n,4)$	$Z(4,n)$	$Z(4,n)$
G_4		$Z(n,4)$	$Z(4,n)$	$Z(4,n)+1$
G_5		$Z(n,4)$	$Z(n,4)$	$Z(4,n)+1$
G_6		$Z(4,n)+\lfloor \frac{n}{2} \rfloor$	$Z(4,n)+\lfloor \frac{n}{2} \rfloor$	$Z(4,n)+\lfloor \frac{n}{2} \rfloor+2$
G_7		$Z(4,n)+\lfloor \frac{n}{2} \rfloor$	$Z(4,n)+\lfloor \frac{n}{2} \rfloor$	$Z(4,n)+\lfloor \frac{n}{2} \rfloor+2$
G_8		$Z(4,n)$	$Z(4,n)+1$	$Z(4,n)+2$
G_9		$Z(4,n)+\lfloor \frac{n}{2} \rfloor$	$Z(4,n)+\lfloor \frac{n}{2} \rfloor$	$Z(4,n)+\lfloor \frac{n}{2} \rfloor+2$
G_{10}		$Z(4,n)+\lfloor \frac{n}{2} \rfloor$	$Z(4,n)+\lfloor \frac{n}{2} \rfloor+1$	$Z(4,n)+\lfloor \frac{n}{2} \rfloor+3$
G_{11}		$Z(4,n)+n$	$Z(4,n)+n+1$	$Z(4,n)+n+4$

Graph H



Graph $H + nK_1$



$$cr(H + nK_1) \leq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n = Z(6,n) + n$$

$$H + nK_1 = H_n = H \cup K_{6,n} = H \cup \big(\bigcup_{i=1}^{n} T^i\big)$$

Theorem $cr(H + nK_1) = 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n \quad for \quad n \geq 1.$

Proof:

- upper bound – Figure

- reverse inequality – induction on n

$n = 1$: $H + K_1$ is a subdivision of K_5

$n = 2$:

r – removal number

$H + 2K_1$:

$$|V| = 8$$
$$|E| = 20 - r$$
$$|F| = (20 - r) - 8 + 2 = 14 - r$$

$$3(14 - r) \leq 2(20 - r) \quad \Rightarrow \quad r \geq 2$$

$n \geq 3$: Let $cr(H_{n-2}) \geq Z(6, n-2) + (n-2)$

Consider – D with $cr_D(H_n) < Z(6, n) + n$

Properties of D: (i) $cr_D(T^i, T^j) \neq 0$ for all $i \neq j$

(ii) there is T^i for which $cr_D(H, T^i) = 0$

(i): If $cr_D(T^{n-1}, T^n) = 0$, then

$$cr_D(H, T^{n-1} \cup T^n) \geq 2 \quad \text{and}$$
$$cr_D(T^i, T^{n-1} \cup T^n) \geq 6 \quad \text{for all } i = 1, 2, \ldots, n-2$$

$$cr_D(H_n) = cr_D(H_{n-2}) + cr_D(T^{n-1} \cup T^n) + cr_D(H_{n-2}, T^{n-1} \cup T^n) \geq$$

$$Z(6, n-2) + (n-2) + 6(n-2) + 2 \geq Z(6, n) + n$$

(ii): $cr_D(H_n) = cr_D(K_{6,n}) + cr_D(H) + cr_D(K_{6,n}, H) \geq$

$Z(6,n) + cr_D(H) + cr_D(K_{6,n}, H)$

$\Rightarrow \qquad cr_D(H) + cr_D(K_{6,n}, H) < n$

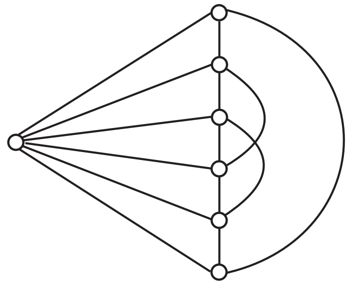
Assume $cr_D(H, T^n) = 0$: $\qquad F^n = H \cup T^n$

$\exists T^i$: $cr_D(F^n, T^i) \leq 3$, $\qquad$ otherwise

$$cr_D(H_n) = cr_D(K_{6,n-1}) + cr_D(F^n) + cr_D(K_{6,n-1}, F^n) \geq$$
$$Z(6, n-1) + 1 + 4(n-1) \geq Z(6,n) + n$$

If $cr_D(C_6(H)) = 0$

s - number of t_i inside $C_6(H)$



Clearly $2s < n$

On the other hand:

$$cr_D(H_n) = cr_D(K_{6,n-1}) + cr_D(F^n) + cr_D(K_{6,n-1}, F^n) \geq$$
$$Z(6, n-1) + 1 + 3s + 5(n - s - 1) \geq$$
$$Z(6, n) + n + (4n - 2s - 6\lfloor\frac{n-1}{2}\rfloor)$$

$$cr_D(H_n) < Z)6, n) + n \quad \Rightarrow \quad 4n - 2s - 6\lfloor\frac{n-1}{2}\rfloor < 0$$

So

$$2s > 4n - 2s - 6\lfloor\frac{n-1}{2}\rfloor \geq n$$

– a contradiction

If $cr_D(C_6(H)) \neq 0$

For some i: $cr_D(F^n, T^i) \leq 3$ and $cr_D(T^i, T^j) \neq 0$

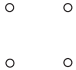
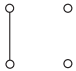
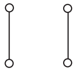
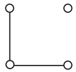
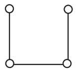
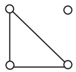
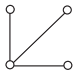
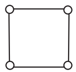
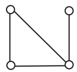
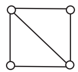
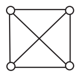
$\Rightarrow$ $cr_D(H, T^i) \leq 2$.

Similar discussion – contradiction □

Theorem $cr(H + P_n) = 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + 1 \quad for \quad n \geq 2.$

Theorem $cr(H + C_n) = 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + 3 \quad for \quad n \geq 3.$

Lemma Let D be a good drawing of $mK_1 + C_n$m $m \geq 2$, $n \geq 3$, in which no edge of C_n is crossed by some edge not belonging to C_n and C_n does not separate the other vertices of the graph. Then, for all $i, j = 1, 2, \ldots, n$, two different subgraphs T^i and T^j cross each other at least $\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$ times.

G_i		$cr(G_i + S_n)$
G_1		$4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 2\left\lfloor\frac{n}{2}\right\rfloor = n(n-1)$
G_2		$4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 2\left\lfloor\frac{n}{2}\right\rfloor = n(n-1)$
G_3		$4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 2\left\lfloor\frac{n}{2}\right\rfloor = n(n-1)$
G_4		$4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 2\left\lfloor\frac{n}{2}\right\rfloor = n(n-1)$
G_5		$4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 2\left\lfloor\frac{n}{2}\right\rfloor = n(n-1)$
G_6		$\leq\ 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + n + \left\lfloor\frac{n}{2}\right\rfloor$
G_7		$4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + n + \left\lfloor\frac{n}{2}\right\rfloor$
G_8		$\leq\ 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + n + \left\lfloor\frac{n}{2}\right\rfloor$
G_9		$\leq\ 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + n + \left\lfloor\frac{n}{2}\right\rfloor$
G_{10}		$\leq\ 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 2n$
G_{11}		$\leq\ 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 2n + \left\lfloor\frac{n}{2}\right\rfloor + 1$

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Thank you for your attention