

Total fractional colorings of graphs with large girth

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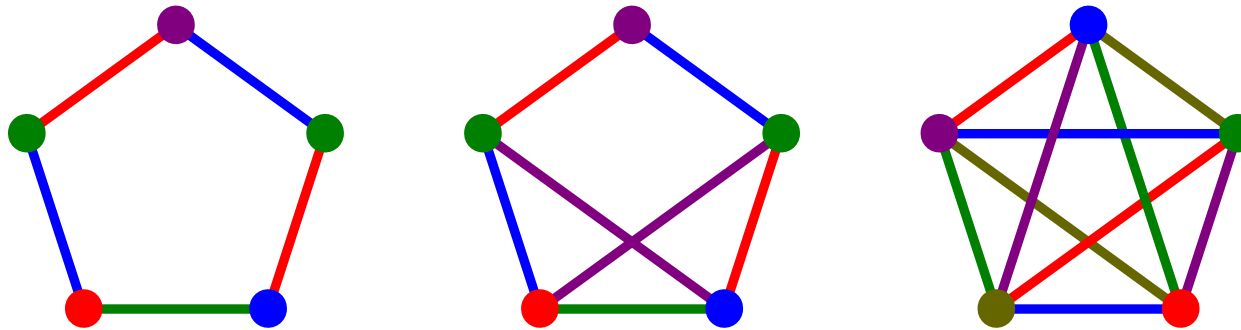
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OVERVIEW

- Total colorings, fractional colorings
- Problem and our results
- Main proof idea

TOTAL COLORINGS

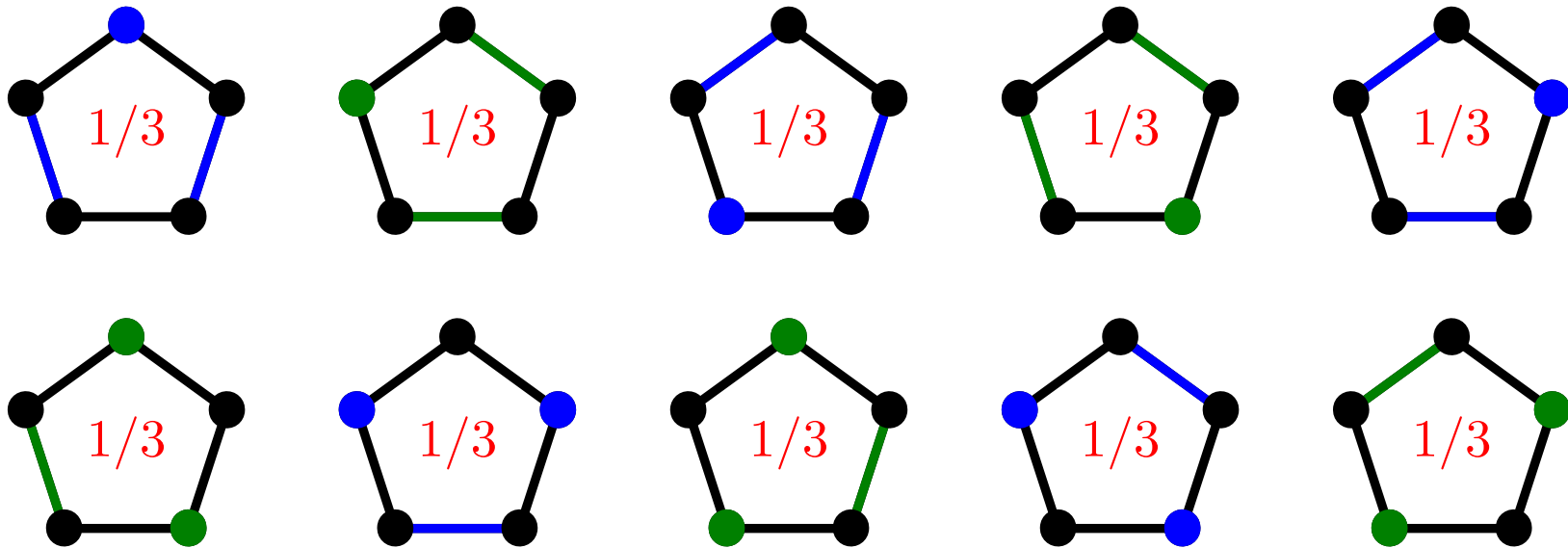


- coloring of vertices and edges
any two adjacent/incident elements must receive distinct colors
- total chromatic number $\chi_t(G)$
- Behzad's conjecture (1965): $\chi_t(G) \leq \Delta + 2$
Molloy and Reed's bound (1998): $\chi_t(G) \leq \Delta + 10^{28}$

FRACTIONAL COLORINGS

- coloring = partitioning into disjoint color classes
- **fractional coloring**
each color class has weight, they need not be disjoint
every element in classes of total weight at least one
minimizing the total weight of all color classes
- total fractional chromatic number $\chi_f(G)$
- weights only zero or one \Rightarrow coloring

EXAMPLE



$$\chi_{t,f} \leq 10/3 \text{ and } \chi_t = 4$$

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TOTAL FRACTIONAL COLORINGS

- Behzad's conjecture is fractionally true

Kilakos and Reed (1993): $\chi_{t,f}(G) \leq \Delta + 2$

- When does the equality hold?

Ito, Kennedy and Reed (2009): only if G is K_{2n} or $K_{n,n}$

- What about large girth?

COLORINGS VS. LARGE GIRTH

- vertex colorings

$\chi_f(G) \leq 2.416$ if G is cubic and has large girth

examples of cubic graphs with large girth with $\chi_f(G) \geq 2.196$

- edge colorings

$\chi'_f(G) = 3$ for cubic bridgeless graphs

$\Delta \leq \chi'_f(G) \leq \Delta + \varepsilon$ for graphs with large girth

REED'S CONJECTURE

Let Δ be an integer. For every $\varepsilon > 0$, there exists g such that every graph G with maximum degree Δ and girth at least g has total fractional chromatic number at most $\Delta + 1 + \varepsilon$.

OUR RESULTS

- Theorem (Kardoš, Král', Sereni):
Let Δ be an integer. For every $\varepsilon > 0$, there exists g such that every graph G with maximum degree Δ and girth at least g has total fractional chromatic number at most $\Delta + 1 + \varepsilon$.
- Theorem (Kaiser, King, Král'):
Let $\Delta \in \{3, 4, 6, 8, \dots\}$. There exists g such that every graph G with maximum degree Δ and girth at least g has total fractional chromatic number equal to $\Delta + 1$.

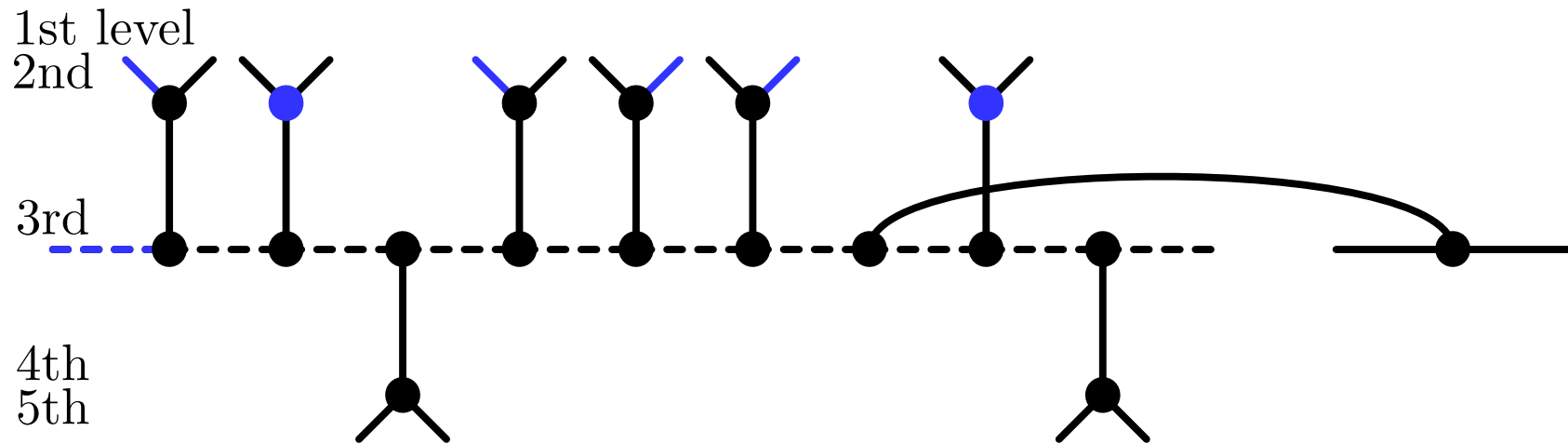
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DECOMPOSITION INTO PATHS

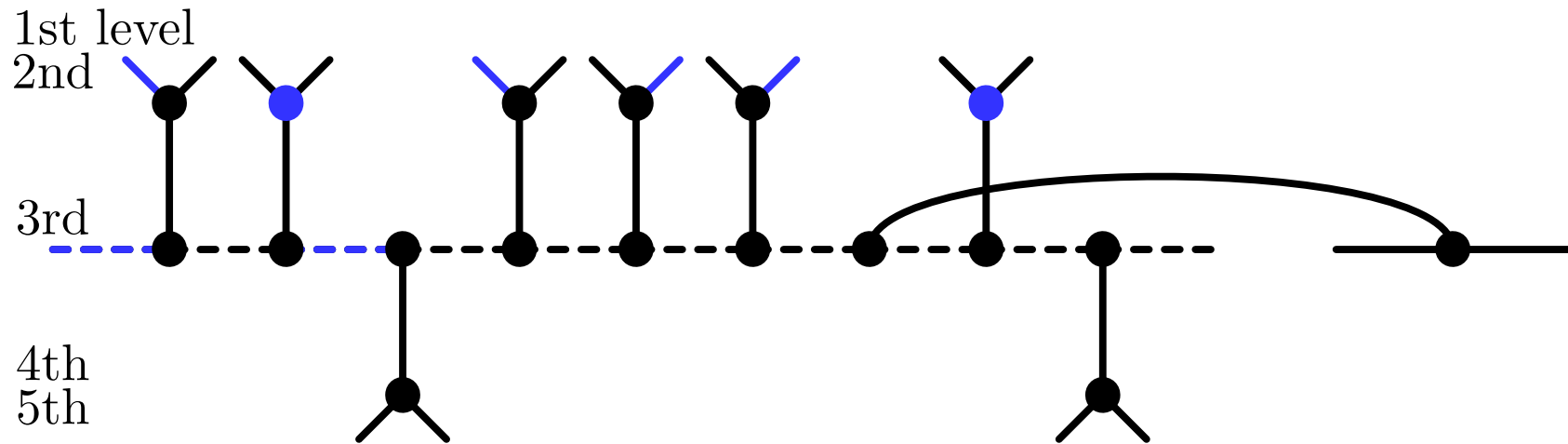
- $\chi_{t,f}(G) \leq \alpha \Leftrightarrow \exists$ probability distribution on independent sets, each element included with probability at least $1/\alpha$
- exposition restricted to cubic bridgeless case for simplicity
 $\alpha = 4 + \varepsilon$
- there exists a distribution on perfect matchings such that each edge included with probability $1/3$
- remove vertices at distance $g/100$ from the complementary 2-factor
collection of paths of length at most $g/100$
each edge included with probability at least $2/3 - 200/g$
each vertex with probability $1 - 100/g$

COLORINGS BY LEVELS



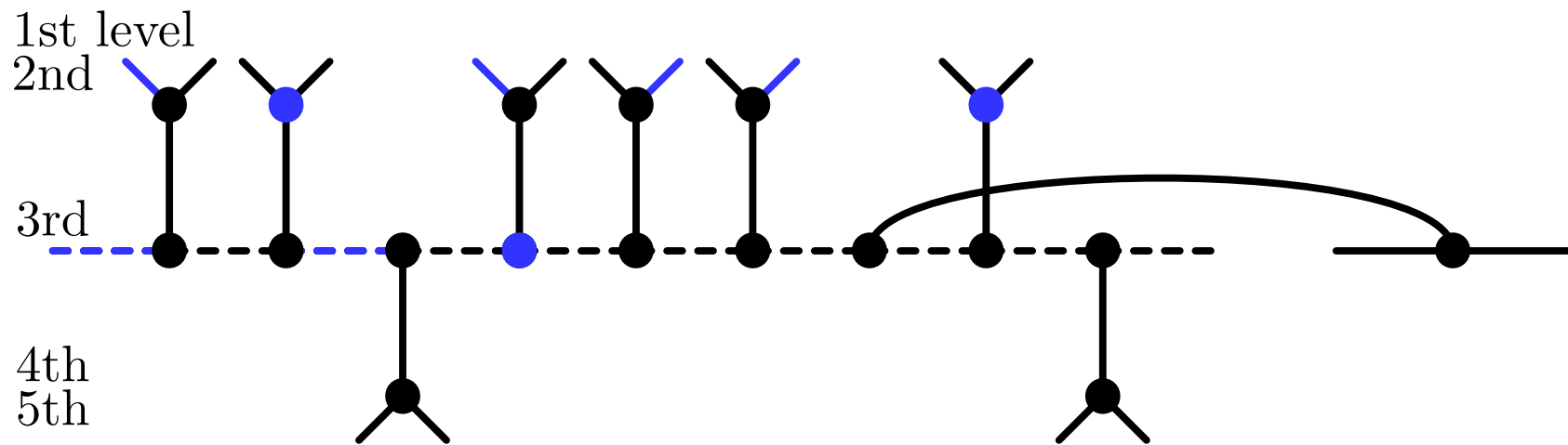
- split paths into 10 levels
- sweep paths in the 1st level, then in the 2nd level, etc.
include vertices and edges greedily into a total independent set
- if a vertex can be included, include it with probability $1 - \xi$

COLORINGS BY LEVELS



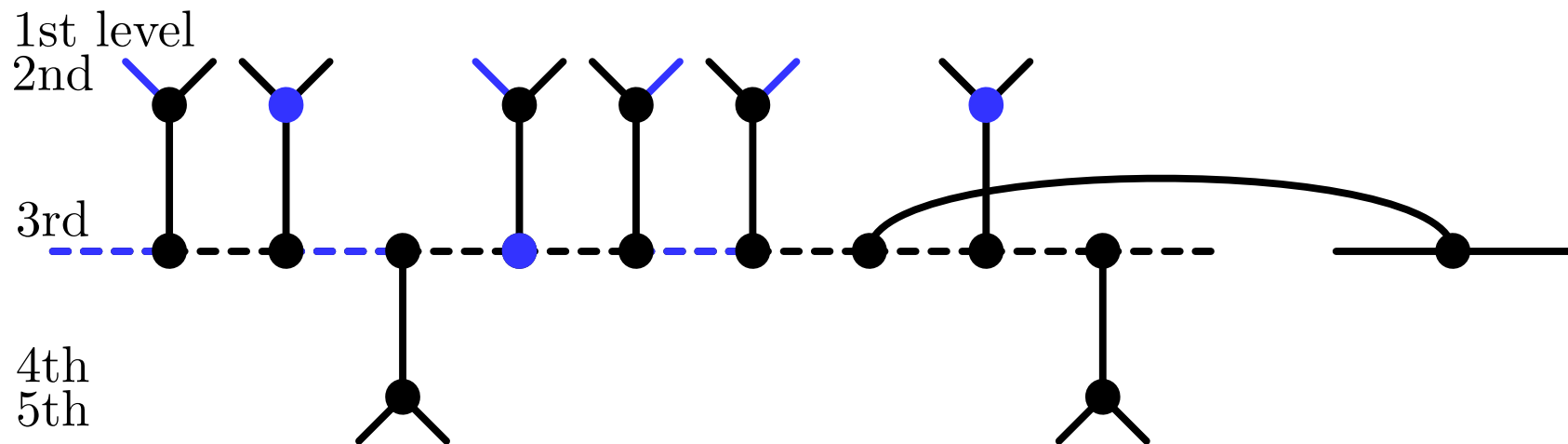
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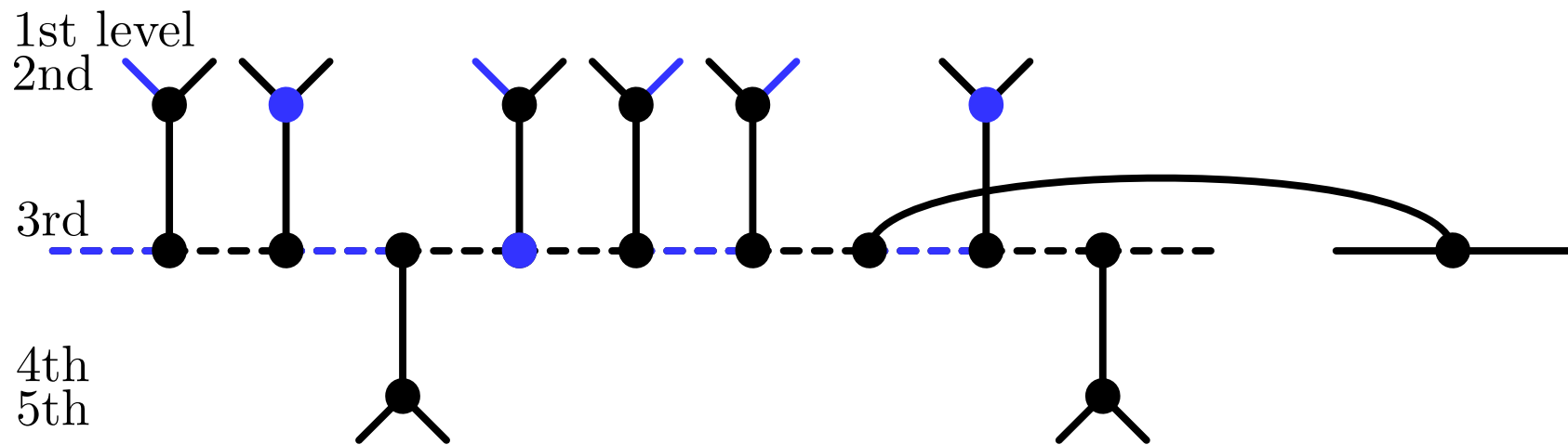
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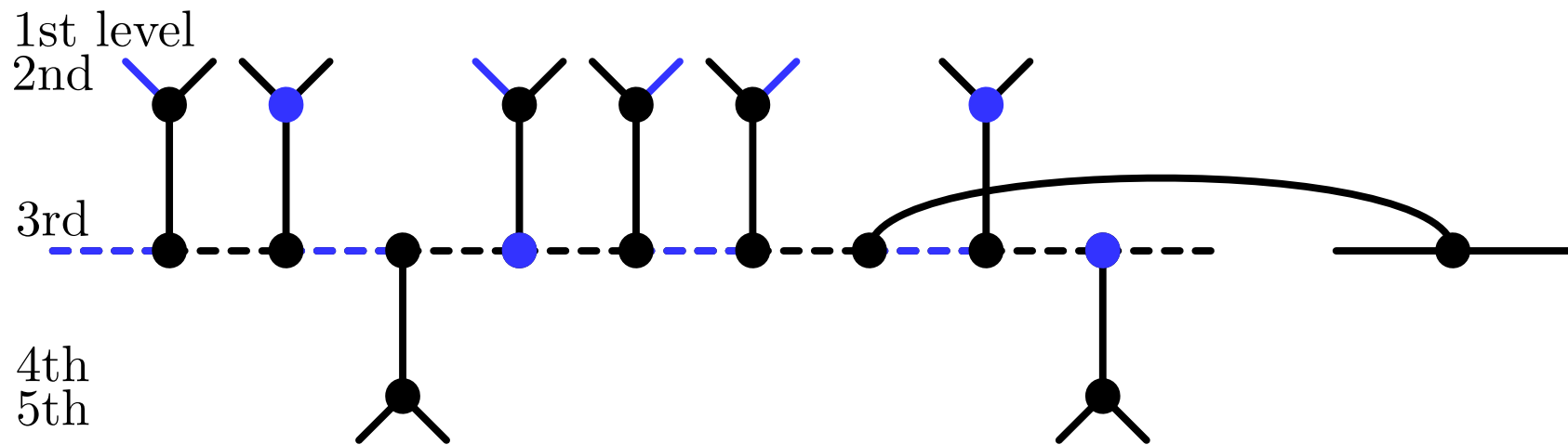
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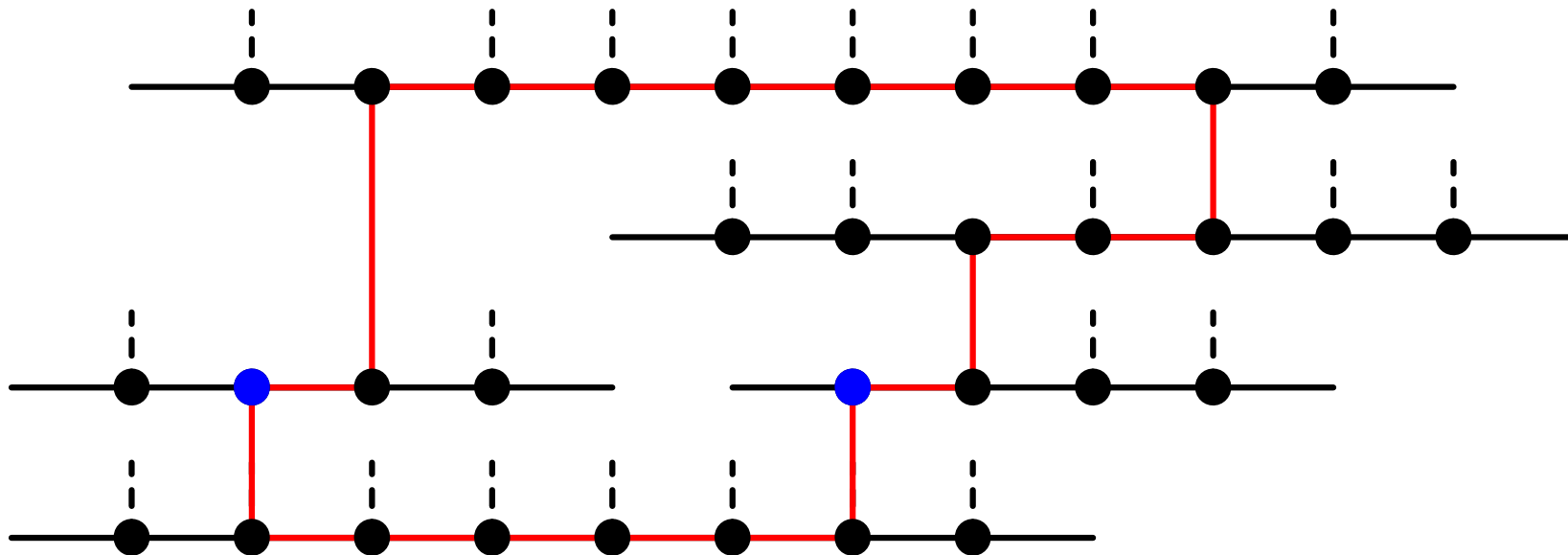
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WHERE DOES THE GIRTH COME IN?

- lengths of paths obtained from cycles
- independence of including mates to the independent set



SUMMARY FOR CUBIC GRAPHS

- levels guarantee mutual independence of neighbors for each path
- for a suitable choice of $\xi > 0$
 - each vertex on a path included with probability $1/4$
 - each edge in a path included with probability $3/8$
- considering the distribution over perfect matchings:
 - each vertex included with probability $1/4 - 25/g$
 - each edge included with probability $1/4 - 75/g$
- a better resolution of coloring conflicts at the ends of the paths needed to obtain $\chi_{t,f} = 4$ for large g

Thank you for your attention!