THE CROSSING NUMBERS OF CARTESIAN PRODUCTS OF SOME SMALL POWER GRAPHS

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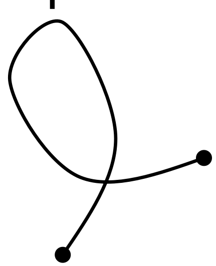
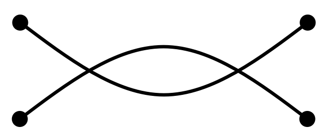
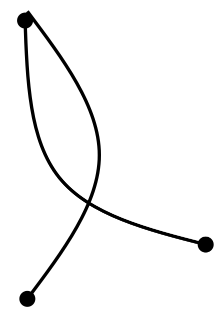
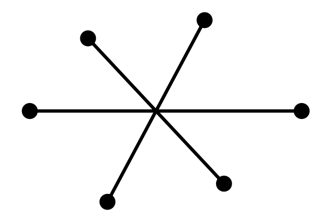
Terms

- <u>Graph</u>: G = (V, E) – simple graph; $E \subseteq \binom{V}{2}$
- <u>Drawing of graph</u>: representation of graph G in the plane

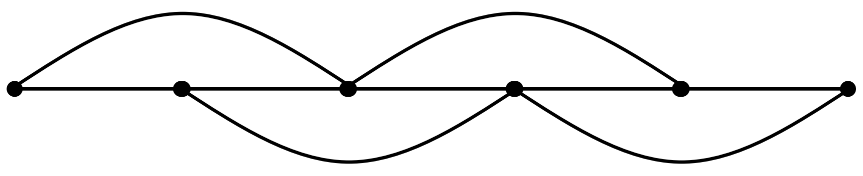
  vertices $\to$ points

  edges $\to$ arcs
- <u>Good drawing of a graph *G*</u>:
  - no edge crosses itself
  - no two edges cross each other more than once
  - no two edges incident with the same vertex cross
  - no more than two edges cross each other in the same point of the plane

Terms

- Crossing: the common interior point of two arcs (not tangential, not endpoint)
- Crossing number of G - cr(G): the minimum possible number of crossings in any drawing of G in the plane. The minimum number of crossings appears in a good drawing of *G*.
- second power of a graph G (power graph G^2):
  - $V(G^2) = V(G)$
  - $E(G^2)$ contains the edges of G and the edges which join two vertices if and only if there is a path of length 2 between them in G.
- Ex:



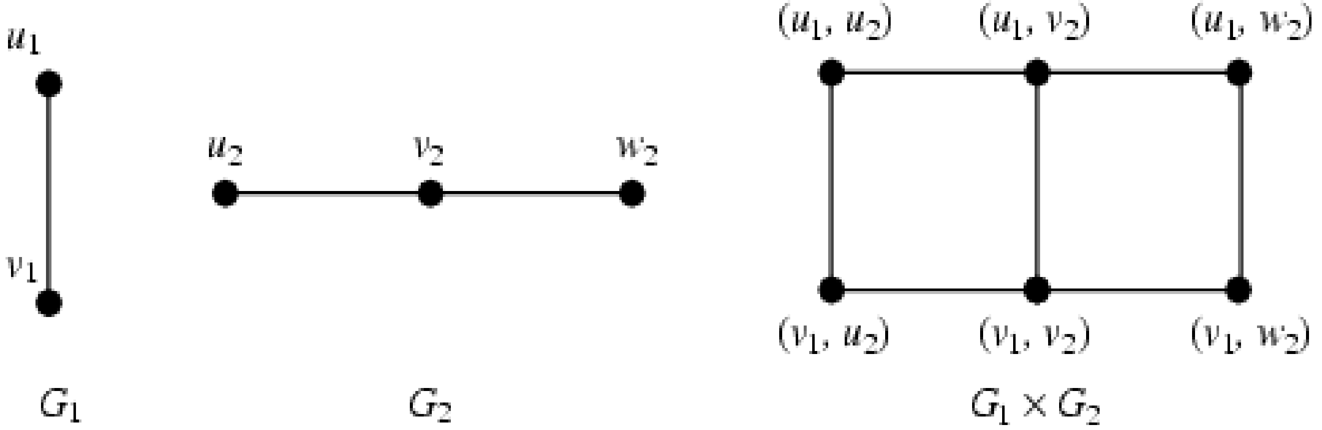
P_5^2

- C_n – cycle on *n* edges
- P_n – path on *n* edges

Terms

<u>Cartesian product $G = (G_1 \times G_2)$ of graphs G_1 and G_2:</u>

- $V(G_1 \times G_2) = V(G_1) \times V(G_2)$
- $E(G_1 \times G_2)$ - any two vertices (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \times G_2$ if and only if either $[u_1 = v_1$ and u_2 is adjacent to v_2 in $G_2]$ or $[u_2 = v_2$ and u_1 is adjacent to v_1 in $G_1]$.



Crossing number of power graphs or Cartesian products with power graphs

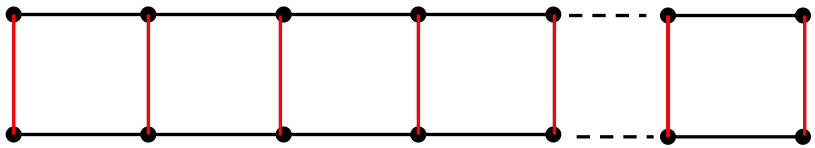
- Patil, Krishnamurthy: *On power graphs with crossing number one*, 1992.
- some results of crossing numbers of power graphs or Cartesian products with power graphs are contained in previous works

  - $P_3^2 = K_4 - e$
  - $P_4^2 = G_{14}$ in [Klešč, 2001, 2005]
  - $C_4^2 = K_4$, $C_5^2 = K_5$
  - $C_6^2 = K_{2,2,2}$ in [Yuan, Huang, 2007]

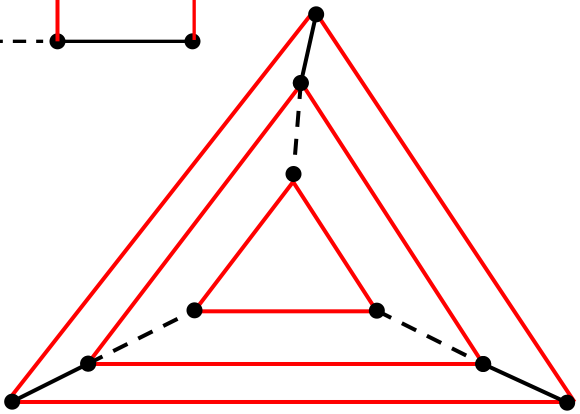
- $cr(C_m^2 \times P_n)$, for $m \leq 8$, ($C_m^2 = C(m,2)$) in [Yuan, Huang, 2008]
- $cr(P_m^2 \times C_n)$, for $m \leq 5$ in [Klešč, Kravecová, 2008]

$cr(P_m^2 \times P_n)$

- $m = 1$; $cr(P_1^2 \times P_n) = cr(P_1 \times P_n) = 0$



- $m = 2$; $cr(P_2^2 \times P_n) = cr(C_3 \times P_n) = 0$



- $m = 3$; $cr(P_3^2 \times P_n) = cr((K_4 - e) \times P_n) = n - 1$; $n \geq 2$

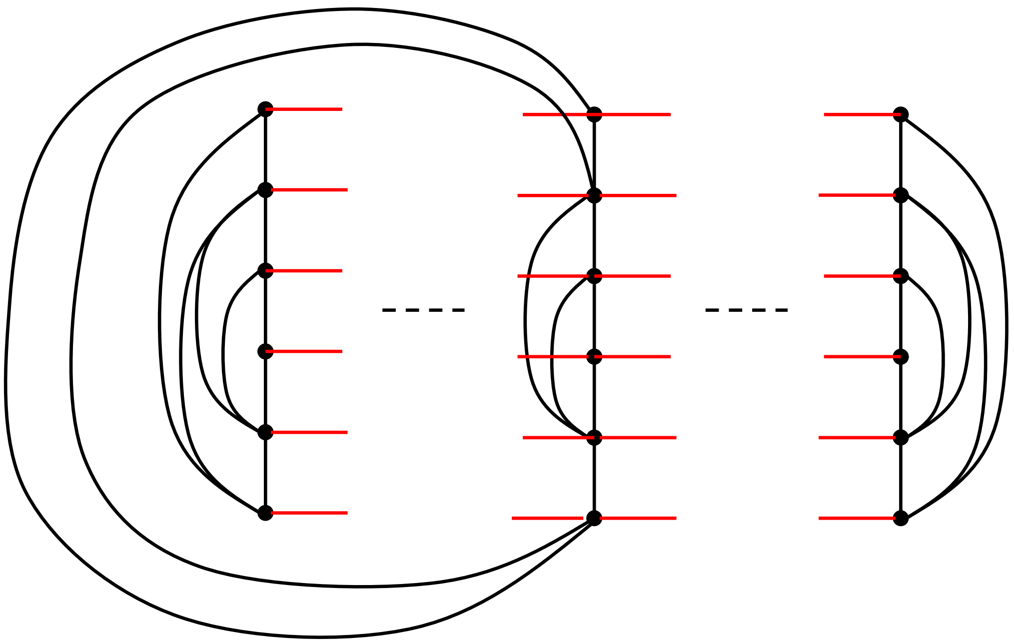
(Klešč, 1994)

- $m = 4$; $cr(P_4^2 \times P_n) = cr(G_{14} \times P_n) = 2(n - 1)$; $n \geq 2$

(Klešč, 2001)

$cr(P_m^2 \times P_n)$ (upper bound)

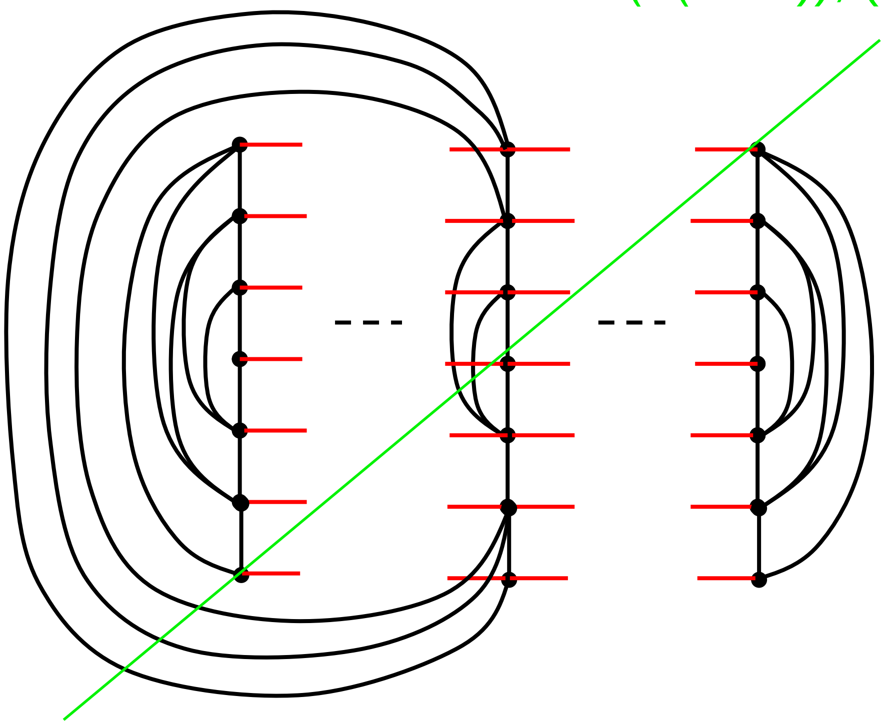
- m = 5; $cr(P_5^2 \times P_n) \leq 4(n - 1)$

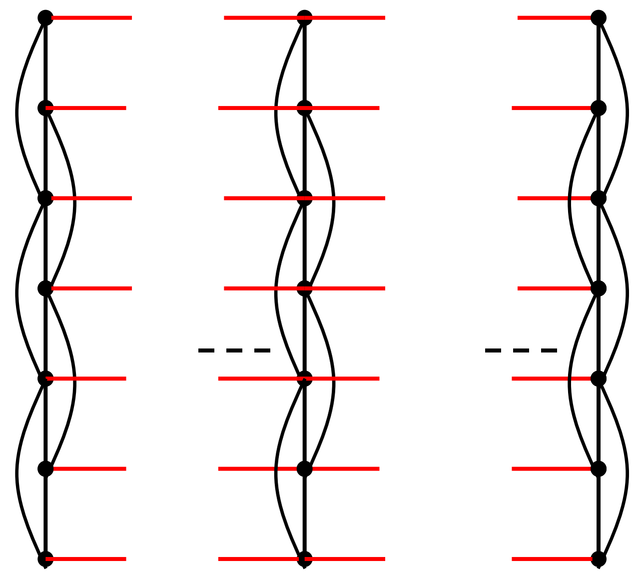


$cr(P_m^2 \times P_n)$ (upper bound)

- m = 6; $cr(P_6^2 \times P_n) \leq 5n - 1$; (better for $n \geq 6$)

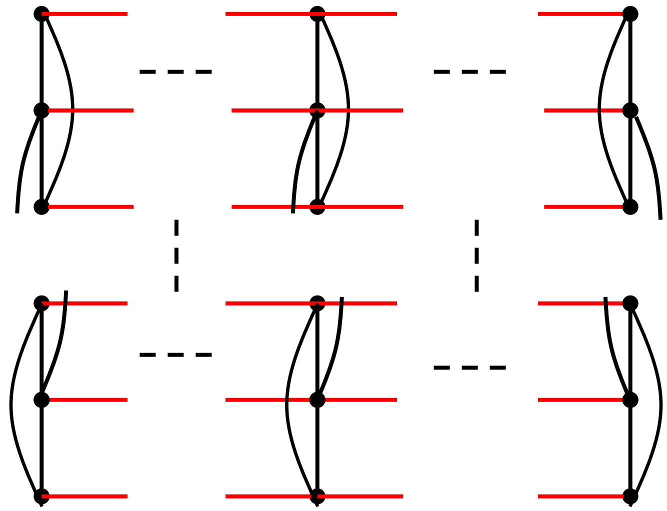
  $\leq (6(n - 1))$; (better for $n < 6$)



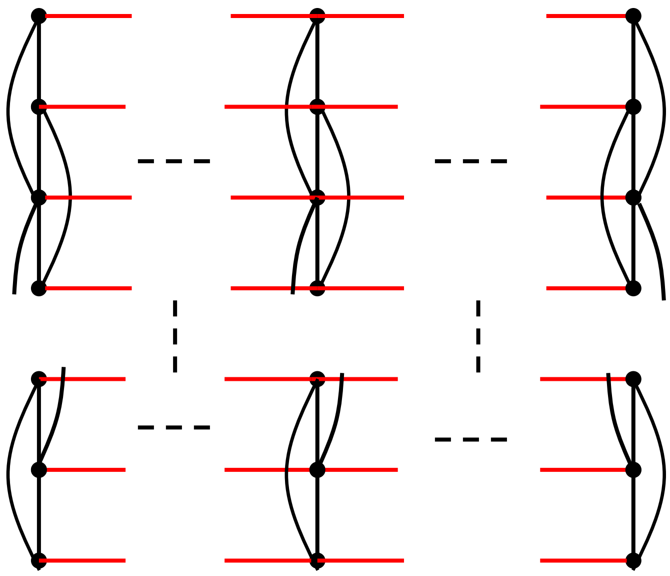


$cr(P_n \times P_m^2)$ (upper bound)

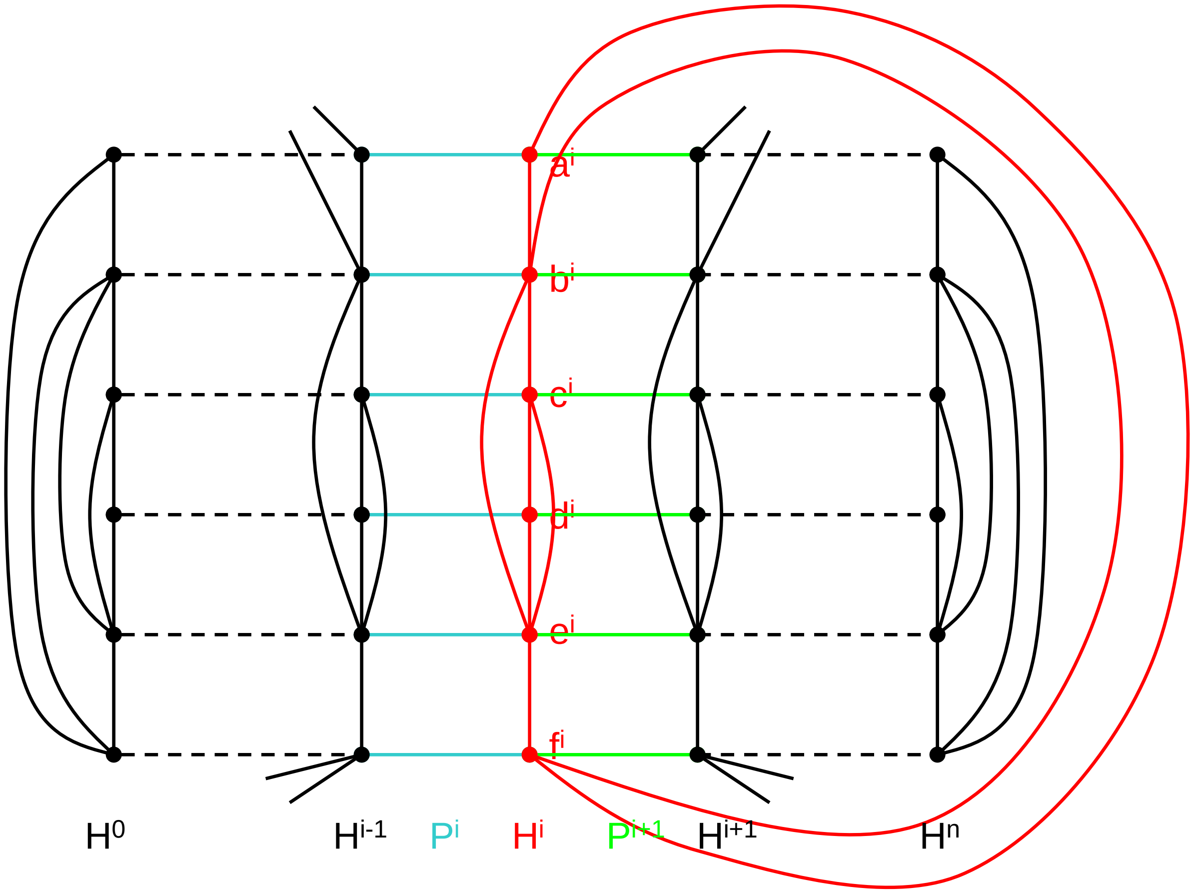
- m – odd , m ≥ 7
- $cr(P_m^2 \times P_n) \leq n.(m - 1)$



- m – even; m ≥ 8
- $cr(P_m^2 \times P_n) \leq n.(m - 1) - 1$

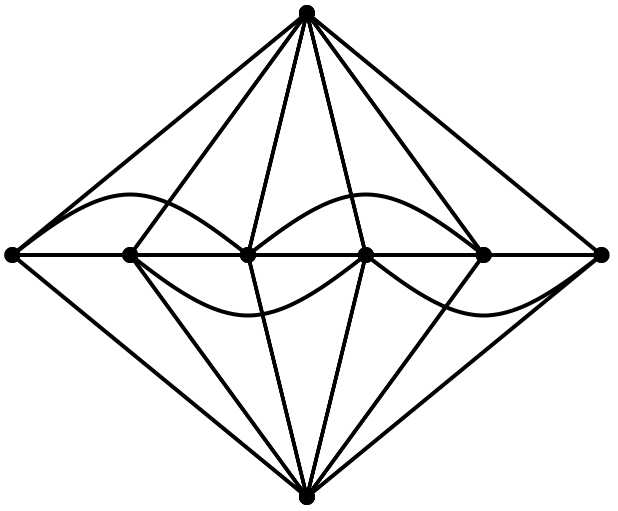


Crossing number of $P_5^2 \times P_n$



Crossing number of $P_5^2 \times P_n$

Lemma 1: cr(Q) = 4; Q is the graph given by next diagram.



Crossing number of $P_5^2 \times P_n$

Lemma 2: If every subgraph H^i; i =1,2,…,n-1 has at most three crossings on its edges in D-good drawing of $P_5^2 \times P_n$, $n \geq 2$, then there are at least 4(n-1) crossings in D.

Proof:

- Two different subgraphs H^i, H^j do not cross each other, in D.
- No H^i is crossed by an edge of P^l for $l \neq i$ and $l \neq i+1$, in D.

- Let Q^i denote the subgraph of $P_5^2 \times P_n$ induced by $V(H^{i-1}) \cup V(H^i) \cup V(H^{i+1})$. Q^i has 27 edges in H^{i-1}, H^i, H^{i+1} and 12 edges in P^i, P^{i+1}.
- Let Q_c^i denote subgraph of Q^i obtained by contracting the edges of H^{i-1} and H^{i+1}.

Crossing number of $P_5^2 \times P_n$

- We define the **force $f(Q^i)$** of Q^i (**$f(Q_c^i)$** of Q_c^i) to be the total number of crossing of the following types in D:

1) a crossing of a some edge in $P^i \cup P^{i+1}$ with some edge in H^i
2) a crossing of a some edge in P^i with a some edge in P^{i+1}
3) a self –intersection in H^i

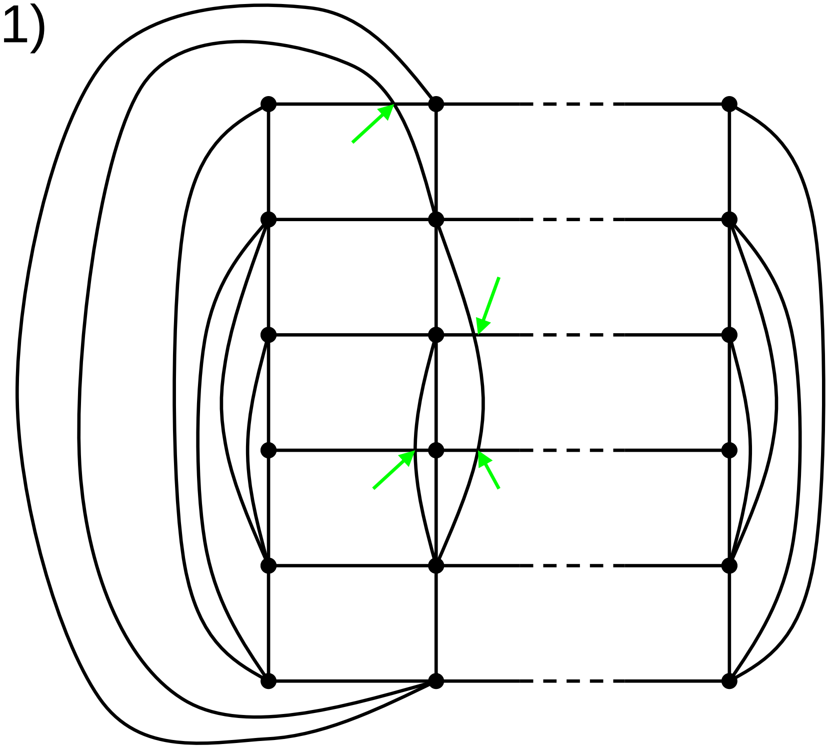
- We define the **total force** of the drawing as the sum of **$f(Q^i)$.**
- In subdrawing D_c^i of Q_c^i induced by D, there are only crossing of types 1), 2), 3).
- Together with previous lemma:
  - $f(Q^i) \geq f(Q_c^i) \geq cr(Q_c^i) = 4$, for i = 1, 2, … , n-1,
  - total force of D is at least 4(n-1)
  - there are at least 4(n-1) crossings, in D.

Crossing number of $P_5^2 \times P_n$

<u>Theorem:</u> $cr(P_5^2 \times P_n) = 4(n - 1)$, $n \geq 1$.

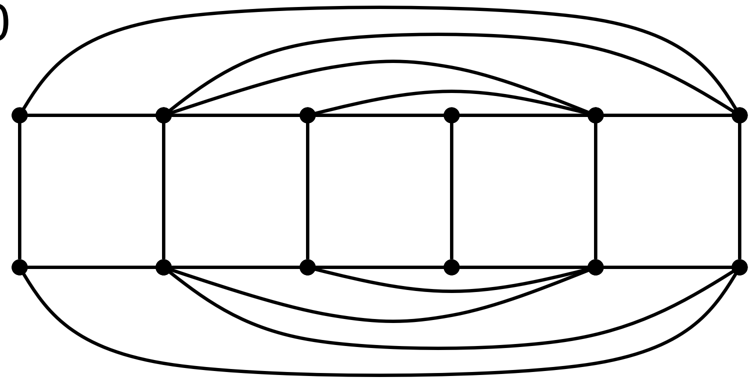
<u>Proof:</u>

- $cr(P_5^2 \times P_n) \leq 4(n - 1)$



Crossing number of $P_5^2 \times P_n$

- $cr(P_5^2 \times P_n) \geq 4(n - 1)$ (by induction with respect n)
  - For $n = 1$; $cr(P_5^2 \times P_1) = 0$



  - Induction hypothesis: It is true for $n = k$, $k \geq 1$. Suppose there is a good drawing of $P_5^2 \times P_{k+1}$ with fewer than $4k$ crossings. We remove all edges of H^i, ($i = 1,\ldots, n-1$) with at least 4 crossings. With respect previous results we obtain a graph homeomorfic to $P_5^2 \times P_k$ or one that contains the subgraph $P_5^2 \times P_k$ and has drawing with fewer than $4(k - 1)$ crossings – contradiction with I. H.

- Conclusion: $cr(P_5^2 \times P_n) = 4(n - 1)$.

THANK YOU FOR YOUR ATTENTION