Closures, Hamiltonian properties and forbidden subgraphs

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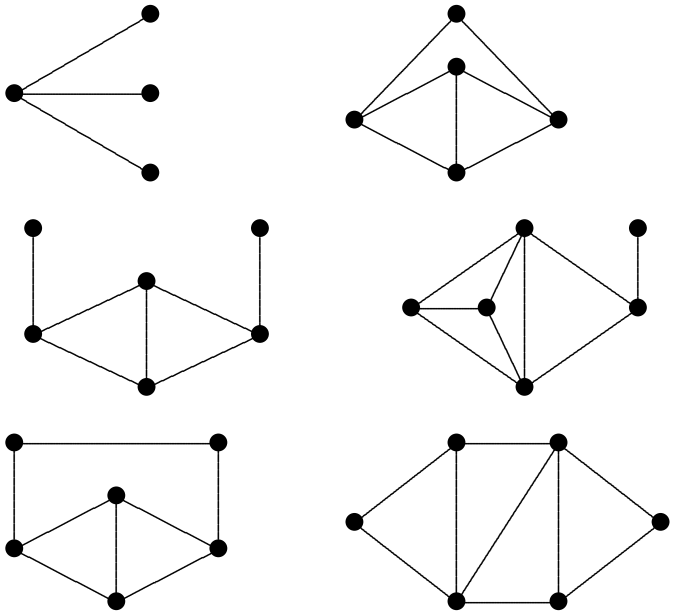
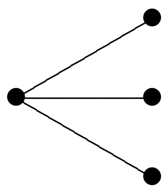
Graph: simple finite undirected, $n = |V(G)|$

Theorem **[Beineke, 1969].** *A graph G is a line graph (of some graph) if and only if G does not contain a copy of any of the following graphs as an* induced *subgraph.*

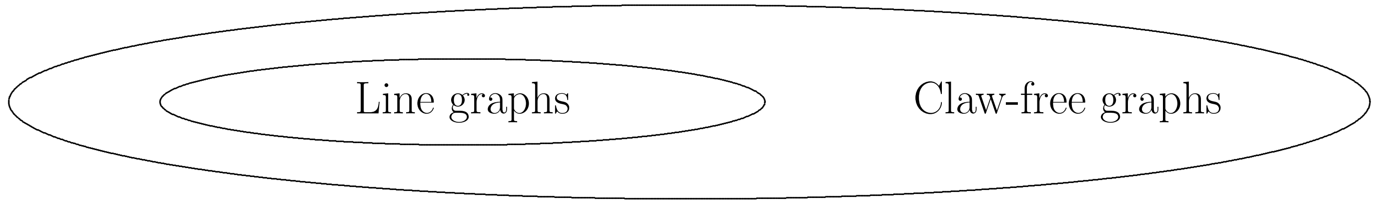


The *claw* $C = K_{1,3}$:



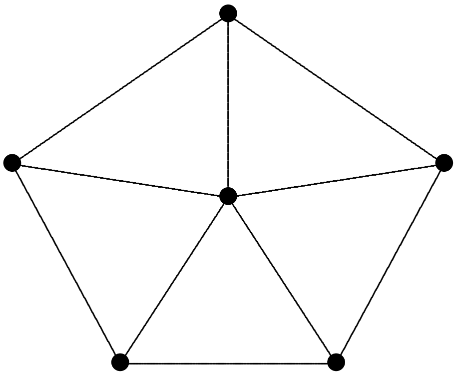
A graph G is *claw-free* if G does not contain a copy of the *claw* $K_{1,3}$ as an induced subgraph

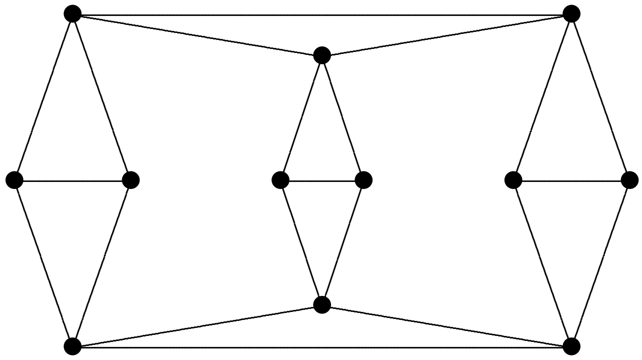
Every line graph is claw-free.



G claw-free

A vertex $x \in V(G)$ is *locally connected* if its neighborhood $N_G(x)$ induces in G a connected graph.





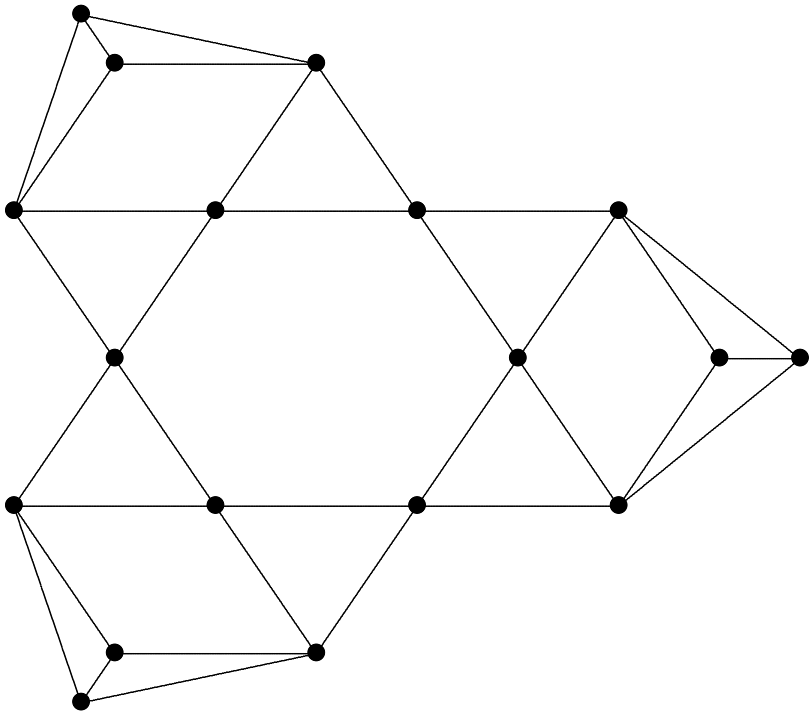
A locally connected vertex with noncomplete neighborhood is called *eligible.*

Let $x \in V(G)$ be an eligible vertex.
The *local completion* of a graph G at x: the graph G'_x with
$V(G'_x) = V(G)$,
$E(G'_x) = E(G) \cup \{xy|\ x, y \in N(x)\}$
"add to the neighborhood of x all missing edges"

$c(G)$: the *circumference* of G (the length of a longest cycle in G)

Theorem [**ZR 1997**]. *Let x be an eligible vertex of a claw-free graph G and let G'_x be the local completion of G at x. Then*
(i) G'_x is claw-free,
(ii) $c(G'_x) = c(G)$.

Corollary. *G'_x is hamiltonian if and only if G is hamiltonian.*



G – a claw-free graph

$\mathrm{cl}(G)$ – the (?) graph obtained from G by recursively performing the local completion operation as long as there is at least one eligible vertex

The graph $\mathrm{cl}(G)$ is called the (claw-free) *closure of G*.

Theorem [**ZR 1997**]. *Let G be a claw-free graph. Then*

(*i*) $\mathrm{cl}(G)$ *is uniquely determined,*

(*ii*) $c(\mathrm{cl}(G)) = c(G)$,

(*iii*) $\mathrm{cl}(G)$ *is the line graph of a triangle-free graph.*

Corollary. *G is hamiltonian if and only if $\mathrm{cl}(G)$ is hamiltonian.*

The closure operation $\mathrm{cl}(G)$:

- turns a claw-free graph into the line graph of a triangle-free graph
- preserves hamiltonicity or non-hamiltonicity

Examples of results which were proved using closure technique:

Theorem [**ZR 1997**]**.** *Every 7-connected claw-free graph is hamiltonian.*

Theorem [**Mulač, Kovářík, ZR 2002**]**.** *Let G be a 2-connected claw-free graph of order $n \geq 153$ with minimum degree*

$$\delta(G) \geq \frac{n+39}{8}.$$

Then either G is hamiltonian, or $G \in \cup_{i=3}^{7} \mathcal{G}_i$.
($\cup_{i=3}^{7} \mathcal{G}_i$ – a family of exceptions).

Tools:

$L(H)$: the *line graph* of a graph H
$H = L^{-1}(G)$: the *line graph preimage* of G

A *dominating closed trail* (abbreviated DCT) in a graph H is a closed trail T such that every edge of H has at least one vertex in T.
(Trail: vertices can be visited several times.)

Theorem [**Harary and Nash-Williams 1965**]**.**
Let H be a graph with at least three edges. Then $L(H)$ is hamiltonian if and only if H contains a DCT.

Observation. *$L(H)$ contains an induced subgraph isomorphic to a graph F if and only if H contains a subgraph isomorphic to the graph $L^{-1}(F)$.*

Conjectures

Conjecture 1 **[Matthews, Sumner 1984].**

Every 4-connected claw-free graph is hamiltonian.

Conjecture 2 **[Thomassen 1986].**

Every 4-connected line graph is hamiltonian.

Theorem **[ZR 1997].**

Conjectures 1 and 2 are equivalent.

Conjecture 3.

Every snark has a dominating cycle.

(Snark: (*i*) cubic, (*ii*) cyclically 4-edge connected, (*iii*) not 3-edge-colorable, (*iv*) no cycle of length $\ell \leq 4$)

Theorem **[Broersma, Fijavž, Kaiser, Kužel, ZR, Vrána 2008].**

Conjectures 1, 2, 3 are equivalent.

G is *Hamilton-connected* if G has a hamiltonian (x, y)-path $\forall x, y \in V(G)$

G is *k-Hamilton-connected* if, for any $X \subset V(G)$ with $|X| = k$, the graph $G - X$ is Hamilton-connected.

Easy: *k-Hamilton-connected* $\Rightarrow$ *$(k+3)$-connected*

Theorem. *The following statements are equivalent:*

(i) *Every snark has a dominating cycle*

(ii) *Every 4-connected claw-free graph is hamiltonian*

(iii) *Every 4-connected line graph is hamiltonian*

(iv) *Every 4-connected line graph is Hamilton-connected*

(v) *Every 4-connected line graph is 1-Hamilton-connected*

(iv): [Kužel, Xiong 2004]

(v): [Kužel, Vrána 2009]

Consider the following two decision problems.

k-HC

Instance: *A graph G.*

Question: *Is G k-Hamilton-connected?*

k-HCL

Instance: *A line graph G.*

Question: *Is G k-Hamilton-connected?*

(i.e., k-HCL is k-HC restricted to line graphs).

Question 1: *Determine the complexity of 1-HCL.*

Known:

HAM

Instance: *A graph G.*

Question: *Does G contain a hamiltonian cycle?*

HAM $\in$ NPC, even if restricted to line graphs.

H-PATH

Instance: *A graph G and distinct vertices $u, v \in V(G)$.*

Question: *Does G contain a hamiltonian (u, v)-path?*

H-PATH $\in$ NPC, even if restricted to line graphs [Bertossi 1981]

H-CONN

Instance: *A graph G.*

Question: *Is G Hamilton-connected?*

H-CONN $\in$ NPC [Dean 1993]

1-H-CONN

Instance: *A graph G.*

Question: *Is G 1-Hamilton-connected?*

1-H-CONN $\in$ NPC [Vrána 2009]

Thus, a common guess would be that probably 1-HCL $\in$ NPC.

Question 2: *Why is Question 1 interesting?*

Recall:

Theorem. *The following statements are equivalent:*

(i) *Every 4-connected line graph is hamiltonian*

(ii) *Every 4-connected line graph is 1-Hamilton-connected*

If the Thomassen's conjecture is true, then:

- A line graph G is 1-Hamilton-connected $\Longleftrightarrow$ G is 4-connected
- 1-HCL is polynomial

Proving the "common guess" 1-HCL $\in$ NPC would mean

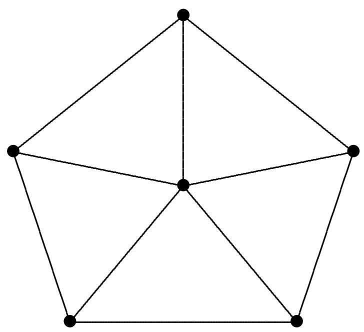
- disproving the Thomassen's conjecture,
- proving the existence of a snark with no dominating cycle,

unless P=NP.

Further cycle and path properties

Theorem **[ZR, Saito, Schelp 1999].**
Let G be a claw-free graph. Then $\mathrm{cl}(G)$ *has a 2-factor with at most k components if and only if G has a 2-factor with at most k components.*

Corollary. *Let G be a claw-free graph. Then* $\mathrm{cl}(G)$ *has a 2-factor if and only if G has a 2-factor.*



No 2-factor with 2 components

$\mathrm{cl}(G)$ complete

$\mathcal{C}$ – a class of graphs

We say that $\mathcal{C}$ *is stable* if $\quad G \in \mathcal{C} \Rightarrow \mathrm{cl}(G) \in \mathcal{C}$.

Examples:

k-connected claw-free graphs

chordal claw-free graphs

$\mathcal{P}$ – a property

$\mathcal{C}$ – a stable class

We say that $\mathcal{P}$ *is stable in* $\mathcal{C}$ if, for any $G \in \mathcal{C}$, $\quad G$ has $\mathcal{P} \Leftrightarrow \mathrm{cl}(G)$ has $\mathcal{P}$.

Examples:

- hamiltonicity
- "having a 2-factor"

are stable properties in the class of k-connected claw-free graphs.

π – a graph invariant

$\mathcal{C}$ – a stable class

We say that π *is stable in* $\mathcal{C}$ if, for any $G \in \mathcal{C}$, $\quad \pi(G) = \pi(\mathrm{cl}(G))$.

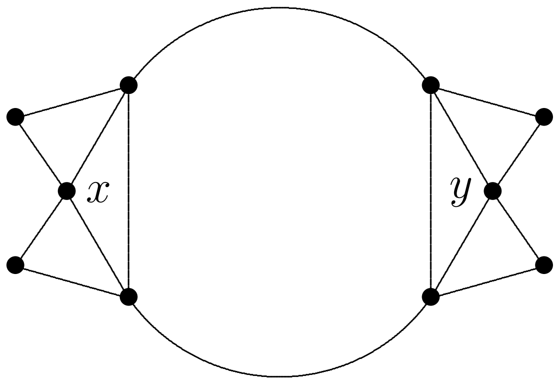
Examples:

- circumference
- mimnimum number of componnets of a 2-factor

are stable invariants in the class of k-connected claw-free graphs.

Having a 2-factor with exactly k components: not stable.

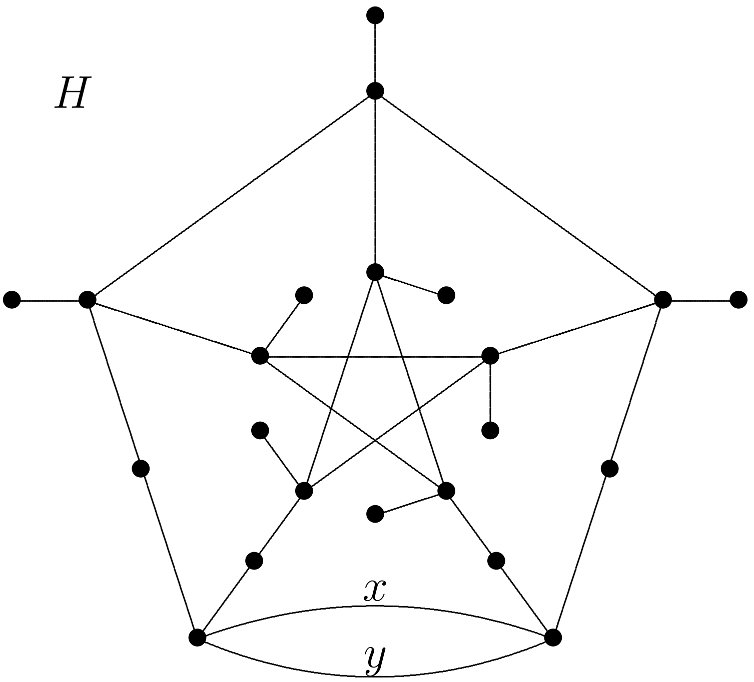
Hamilton-connectedness:



NOT STABLE

Hamilton-connectedness is not stable in 2-connected claw-free graphs.

$G = L(H)$:



NOT STABLE in 3-connected claw-free graphs

BUT: every 7-connected claw-free graph is Hamilton-connected $\Rightarrow$
Hamilton-connectedness IS STABLE in 7-conected claw-free graphs.

Hamilton-connectedness is:

not stable in 3-conected claw-free graphs.

stable in 7-conected claw-free graphs.

Property / invariant	Stable	Connectivity
Circumference	YES	1
Hamiltonicity	YES	1
Having a 2-factor with $\leq k$ components	YES	1
Minimum number of components in a 2-factor	YES	1
Having a cycle cover with $\leq k$ cycles	YES	1
Minimum number of cycles in a cycle cover	YES	1
(Vertex) pancyclicity	NO	any $\kappa \geq 2$
(Full) cycle extendability	NO	any $\kappa \geq 2$
Length of a longest path	YES	1
Traceability	YES	1
Having a path factor with $\leq k$ components	YES	1 [Ishizuka]
Minimum number of components in a path factor	YES	1 [Ishizuka]
Having a path cover with $\leq k$ paths	YES	1 [Ishizuka]
Minimum number of paths in a path cover	YES	1 [Ishizuka]
Homogeneous traceability	NO	3
	???	$4 \leq \kappa \leq 5$
	YES	6
Hamilton-connectedness	NO	3
	???	$4 \leq \kappa \leq 6$
	YES	7
Having a P_3-factor	NO	1
	YES	2 [Kaneko et al.]
Flower property	YES	1
Hamiltonian index	YES	1
2-factor index	YES	1 [Xiong, Saito]
Minimum number of components in a 2-factor in k-th iterated line graph	YES	1 [Xiong, Saito]
Supereulerian index	YES	1 [Xiong, Li]
Having hamiltonian prism	YES	1 [Čada]

Closure for 2-factors

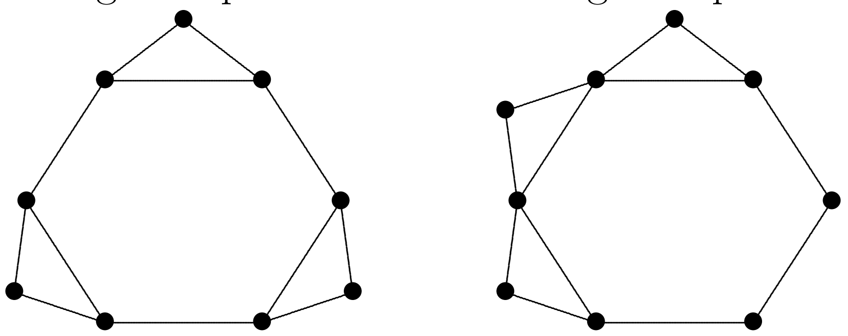
C_k – a cycle of even length $k \geq 4$.

$e_1, e_2 \in E(G)$ are *antipodal in* C_k, if they are at maximum distance in C_k (i.e., $\text{dist}_{C_k}(e_1, e_2) = k/2 - 1$),

C_k is *edge-antipodal in* G, abbreviated EA, if $\min\{\omega_G(e_1), \omega_G(e_2)\} = 2$ for any two antipodal edges $e_1, e_2 \in E(C)$.

($\omega_G(e)$ – the largest order of a clique containing e)

Edge-antipodal   Edge-antipodal   Not edge-antipodal

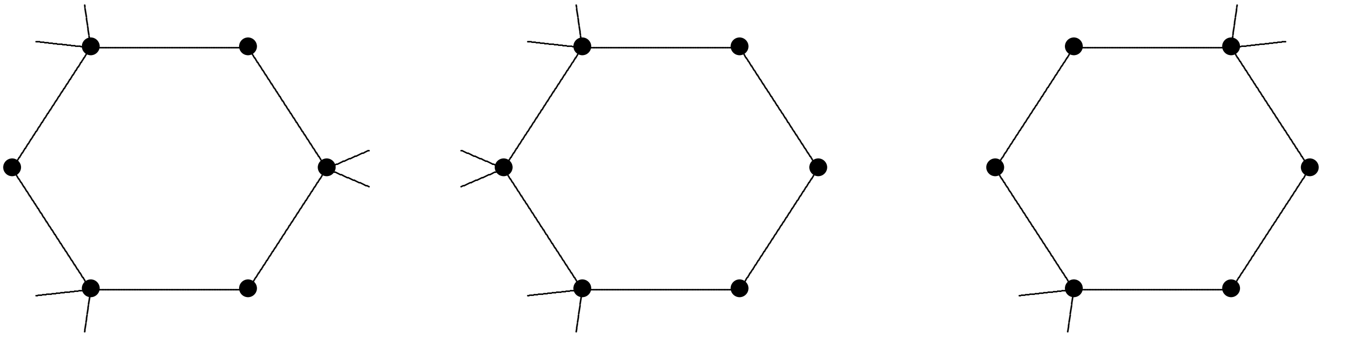


Analogously:

$x_1, x_2 \in V(C_k)$ are *antipodal in* C_k if they are at maximum distance in C_k (i.e. $\text{dist}_{C_k}(x_1, x_2) = k/2$),

C_k is *vertex-antipodal in* G, abbreviated VA, if $\min\{d_G(x_1), d_G(x_2)\} = 2$ for any two antipodal vertices $x_1, x_2 \in V(C_k)$.

Vertex-antipodal   Vertex-antipodal   Not vertex-antipodal



A vertex $x \in V(G)$ is *2f-eligible*, if x satisfies one of the following:

(*i*) $x \in \mathrm{EL}(G)$,

(*ii*) $x \notin \mathrm{EL}(G)$ and x is in an induced cycle of length 4 or 5 or in an induced EA-cycle of length 6.

The set of all 2f-eligible vertices of G will be denoted $\mathrm{EL}^{2f}(G)$.

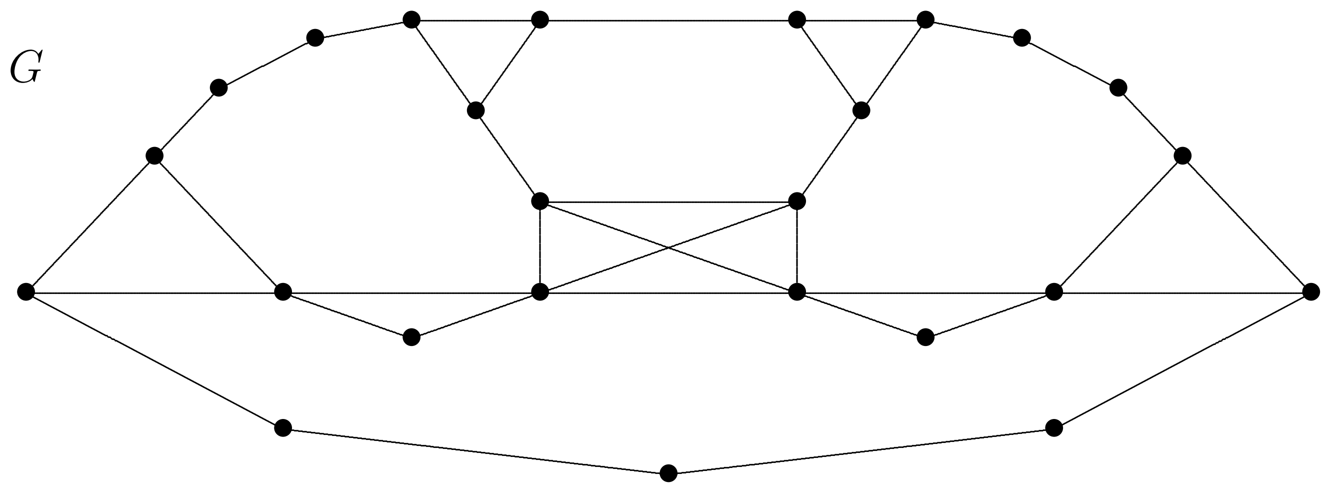
We say that a graph $\mathrm{cl}^{2f}(G)$ is a *2-factor-closure* (abbreviated 2f-closure) of a claw-free graph G, if there is a sequence of graphs $G_1, \ldots, G_k$ such that

(*i*) $G_1 = G$,

(*ii*) $G_{i+1} = (G_i)^*_{x_i}$ for some $x_i \in \mathrm{EL}^{2f}(G_i)$, $i = 1, \ldots, k-1$,

(*iii*) $G_k = \mathrm{cl}^{2f}(G)$ and $\mathrm{EL}^{2f}(G_k) = \emptyset$.

Thus, the 2f-closure of is obtained by recursively repeating the local completion operation at 2f-eligible vertices, as long as this is possible.



Theorem **[ZR, Xiong, Yoshimoto 2009].**
Let G be a claw-free graph. Then

(i) *the closure* $\mathrm{cl}^{2f}(G)$ *is uniquely determined,*

(ii) *there is a graph H such that*

(α) $L(H) = \mathrm{cl}^{2f}(G)$,

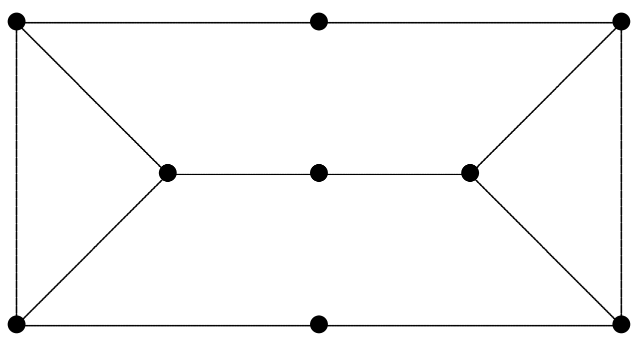
(β) $g(H) \geq 6$,

(γ) *H does not contain any vertex-antipodal cycle of length 6,*

(iii) *G has a 2-factor if and only if* $\mathrm{cl}^{2f}(G)$ *has a 2-factor.*

Corollary. *Let G be a claw-free graph in which every locally disconnected vertex is in an induced cycle of length 4 or 5, or in an induced EA-C_6. Then G has a 2-factor.*

Note: $\mathrm{cl}^{2f}(G)$ does not preserve
– (non)-hamiltonicity
– minimum number of components of a 2-factor



No 2-factor
Every vertex in a cycle of length at most 6
Thus, the antipodality condition cannot be omitted

Application:

Theorem [**Faudree, Faudree, ZR 2008**].
Let X and Y be connected graphs with $X, Y \not\simeq P_3$, and let G be a 2-connected graph of order $n \geq 10$. Then, G being XY-free implies that G has a 2-factor if and only if, up to the order of the pairs, either

(i) $\{X, Y\} = \{K_{1,4}, P_4\}$, *or*

(ii) $X = K_{1,3}$ *and* Y *is an induced subgraph of at least one of the graphs* P_7, $B_{1,4}$ *or* $N_{1,1,3}$.

Theorem [**ZR, Saburov 2009**].
If G is 2-connected and CP_7-free or $CB_{1,4}$-free, then $\mathrm{cl}^{2f}(G)$ is $CN_{1,1,3}$-free.

Corollary. *Let G be a 2-connected XY-free graph of order $n \geq 10$, where X, Y is a pair of connected graphs such that G being XY-free implies G has a 2-factor. Then either*

(i) $\{X, Y\} = \{K_{1,4}, P_4\}$, *or*

(ii) $X = K_{1,3}$ *and* $\mathrm{cl}^{2f}(G)$ *is* $N_{1,1,3}$*-free.*

Closure for Hamilton-connectedness

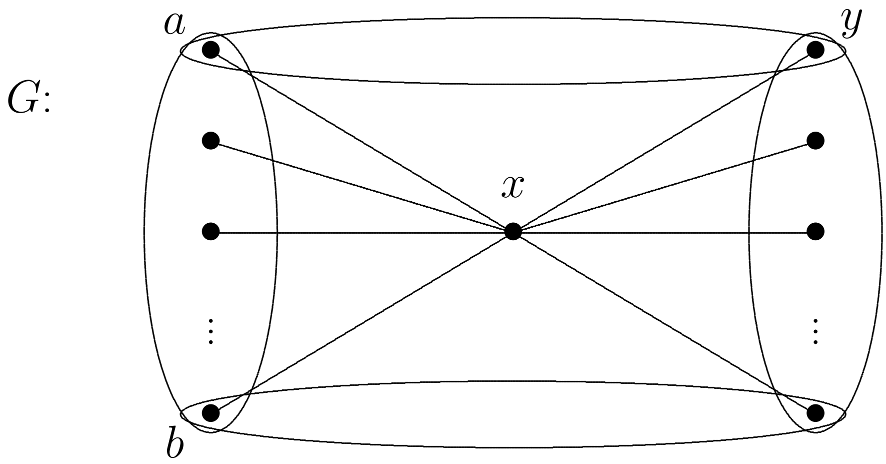
$\mathrm{cl}_k(G)$: closing only neighborhoods of locally k-connected vertices.

Bollobás, Riordan, ZR., Saito, Schelp, 1999:

Theorem. *Hamilton-connectedness is stable under* $\mathrm{cl}_3(G)$.

Conjecture. *Hamilton-connectedness is stable under* $\mathrm{cl}_2(G)$.

Example.



- $\langle N(x)\rangle$ 2-connected
- No hamiltonian (a,b)-path
- There is a hamiltonian (a,b)-path in G'_x.

Property "Having a hamiltonian (a,b)-path" is not stable under $\mathrm{cl}_2(G)$.

BUT: G'_x has no hamiltonian (a,y)-path.

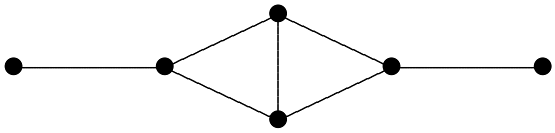
Thus: G'_x is not Hamilton-connected.

Theorem [ZR., Vrána, 2009].

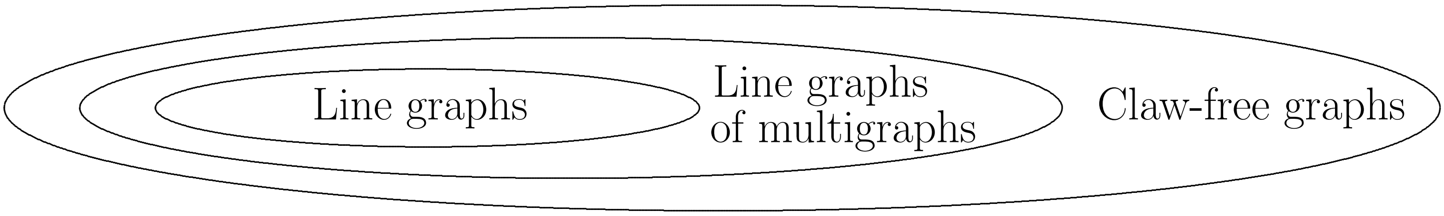
Hamilton-connectedness is stable under $\mathrm{cl}_2(G)$.

What is the structure of $\mathrm{cl}_2(G)$?

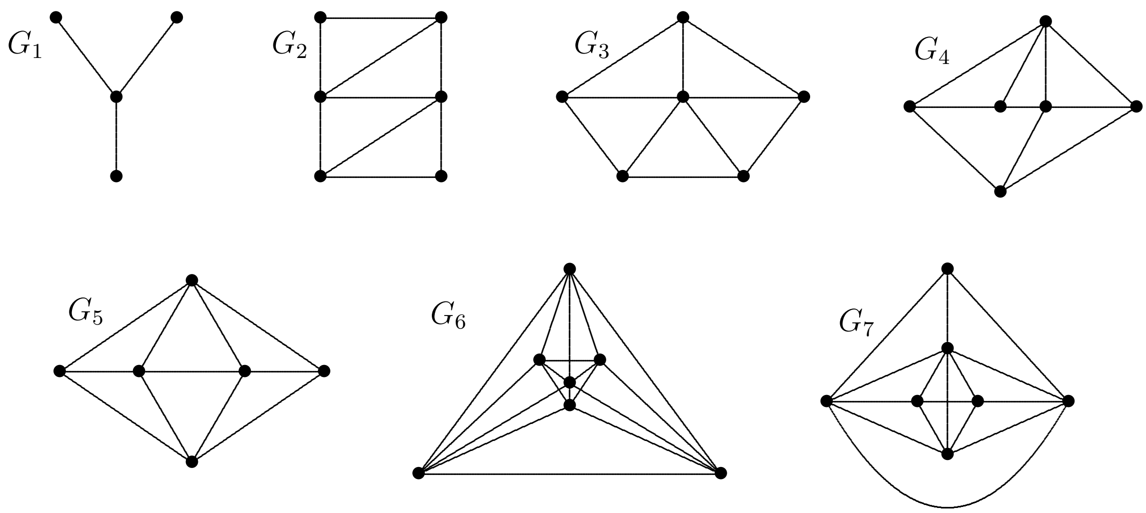
Not a line graph:



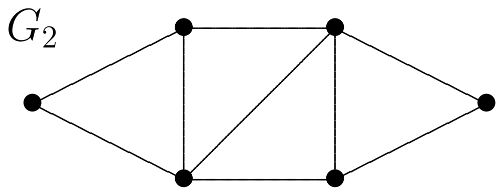
Line graph of a multigraph?

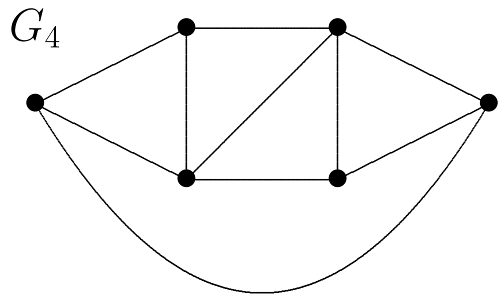


Theorem [Hemminger 1971; Bermond, Meyer, 1973].
A graph G is a line graph of a multigraph *if and only if G does not contain a copy of any of the following graphs as an* induced *subgraph.*



Thus, in $\mathrm{cl}_2(G)$, only G_2 or G_4 can remain.

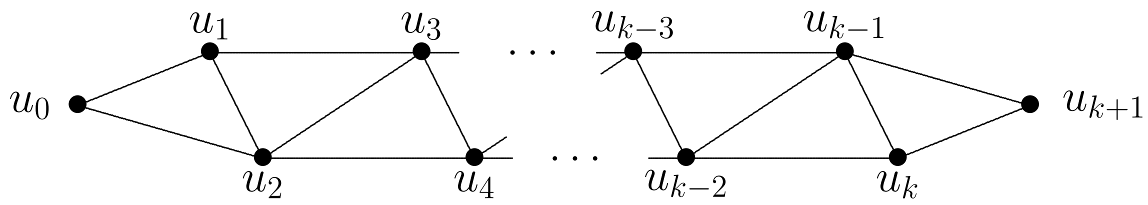




“Multigraph closure” $\mathrm{cl}^M(G)$ of a graph G:
Recursively perform $\mathrm{cl}_2(G)$ and closing specified vertices in copies of G_2 or G_4, as long as there is something to do.

$J = u_0u_1 \dots u_{k+1}$ – a walk in G. We say that J is *good in* G, if

- $k \geq 4$,
- $J^2 \subset G$,
- for any i, $0 \leq i \leq k-4$, $\langle\{u_i, u_{i+1}, \dots, u_{i+5}\}\rangle_G \simeq G_2$ or G_4.

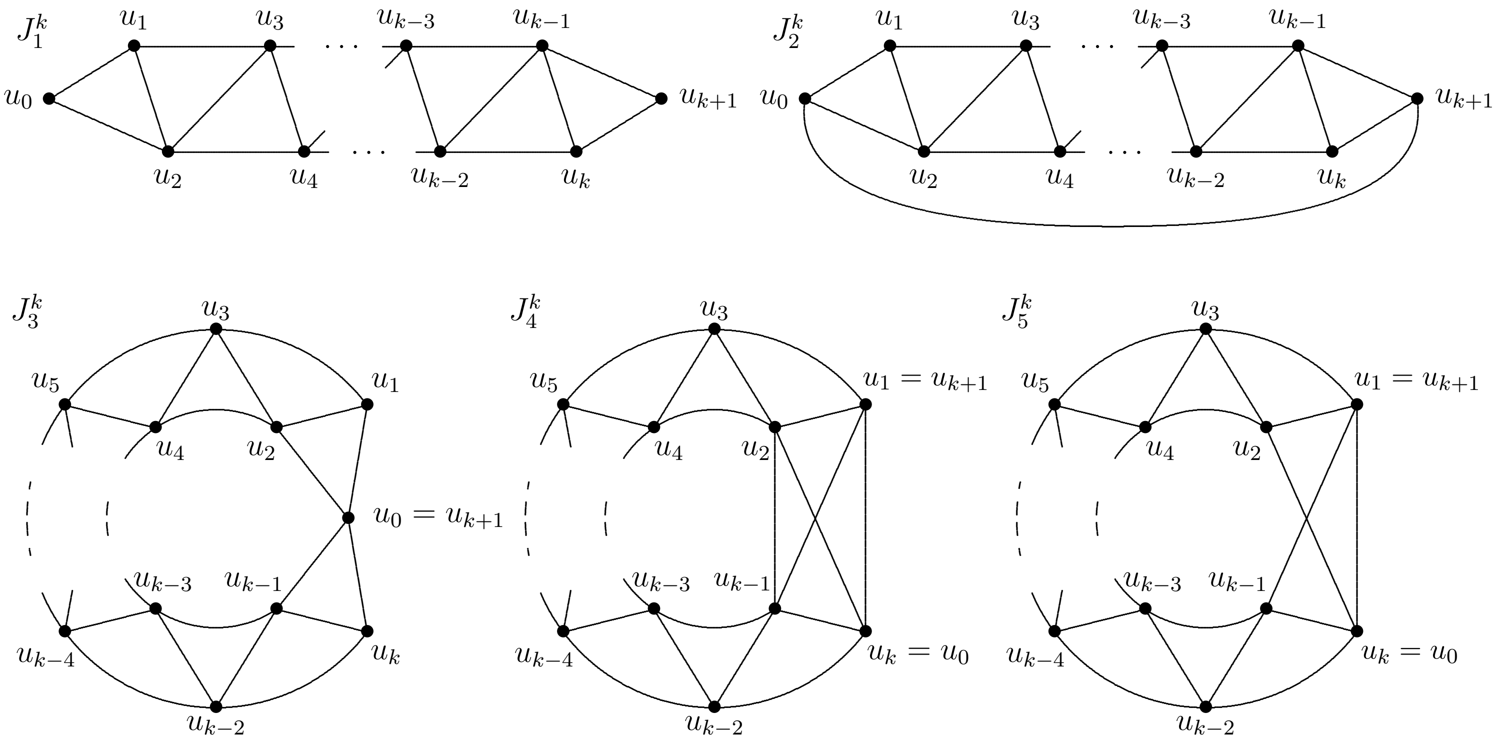


G – claw-free 2-closed, $J = u_0u_1 \dots u_{k+1}$ – a good walk in G. Then

(i) $J = u_1 \dots u_k$ is a path,

(ii) if $k \geq 5$, then $d_G(u_i) = 4$, $3 \leq i \leq k-2$,

(iii) $\langle N_G[u_1] \setminus \{u_3\}\rangle_G = \langle N_G[u_2] \setminus \{u_3, u_4\}\rangle_G$ is a clique.



J_1^k
u_0 u_1 u_2 u_3 u_4 u_{k-3} u_{k-2} u_{k-1} u_k u_{k+1}
J_2^k
u_0 u_1 u_2 u_3 u_4 u_{k-3} u_{k-2} u_{k-1} u_k u_{k+1}
J_3^k
u_3 u_5 u_1 u_4 u_2 $u_0 = u_{k+1}$ u_{k-3} u_{k-1} u_{k-4} u_k u_{k-2}
J_4^k
u_3 u_5 $u_1 = u_{k+1}$ u_4 u_2 u_{k-3} u_{k-1} u_{k-4} $u_k = u_0$ u_{k-2}
J_5^k
u_3 u_5 $u_1 = u_{k+1}$ u_4 u_2 u_{k-3} u_{k-1} u_{k-4} $u_k = u_0$ u_{k-2}

Let G be a claw-free graph, and let $\mathrm{cl}^M(G)$ be a graph constructed by the following algorithm.

1. Set $G_1 = \mathrm{cl}_2(G)$, $i := 1$.
2. If G_i contains a good walk, then
   (a) choose a maximal good walk $J = u_0 u_1 \ldots_{k+1}$,
   (b) let G_i' be the local completion of G_i at u_1 and G_i'' be the local completion of G_i' at u_k,
   (c) set $G_{i+1} = \mathrm{cl}_2(G_i'')$, $i := i + 1$,
   (d) go to (2).
3. Set $\mathrm{cl}^M(G) = G_i$.

The graph $\mathrm{cl}^M(G)$ will be called (the) M*-closure* of G.

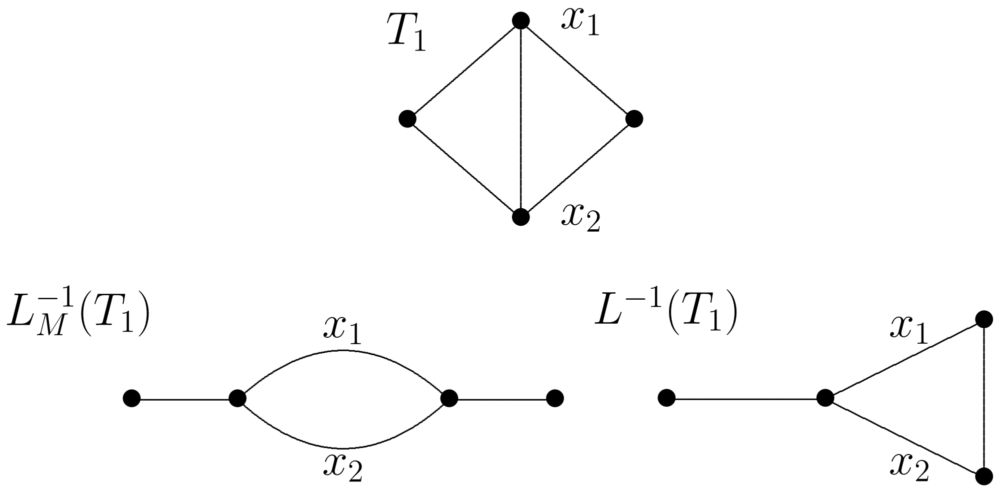
Theorem [ZR., Vrána, 2009].
Let G be a claw-free graph. Then
(*i*) $\mathrm{cl}^M(G)$ *is uniquely determined,*
(*ii*) *there is a multigraph H such that* $\mathrm{cl}^M(G) = L(H)$,
(*iii*) *G is Hamilton-connected if and only if* $\mathrm{cl}^M(G)$ *is Hamilton-connected.*

Hamilton-connectedness is stable under $\mathrm{cl}^M(G)$.

Recall: Every 7-connected line graph of a multigraph is Hamilton-connected [Zhan, 1991].

Corollary [ZR., Vrána, 2009].
Every 7-connected claw-free graph is Hamilton-connected.

Drawback: there can be multigraphs H_1, H_2 such that $H_1 \not\simeq H_2$ but $L(H_1) \simeq L(H_2)$ (i.e., the "preimage" is not uniquely determined).

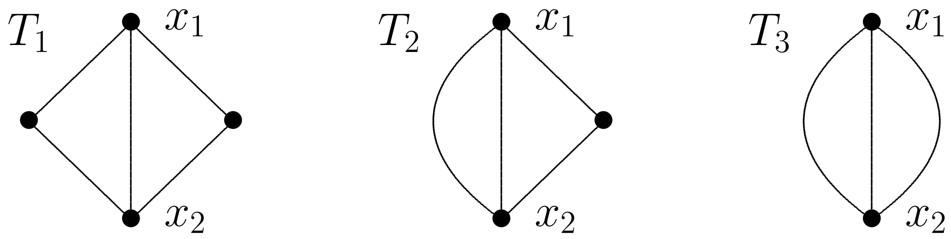


Theorem. *Let G be a connected line graph of a multigraph. Then there is, up to an isomorphism, a uniquely determined multigraph $H = L_M^{-1}(G)$ such that a vertex $e \in V(G)$ is simplicial in G if and only if the corresponding edge $e \in E(H)$ is a pendant edge in H.*

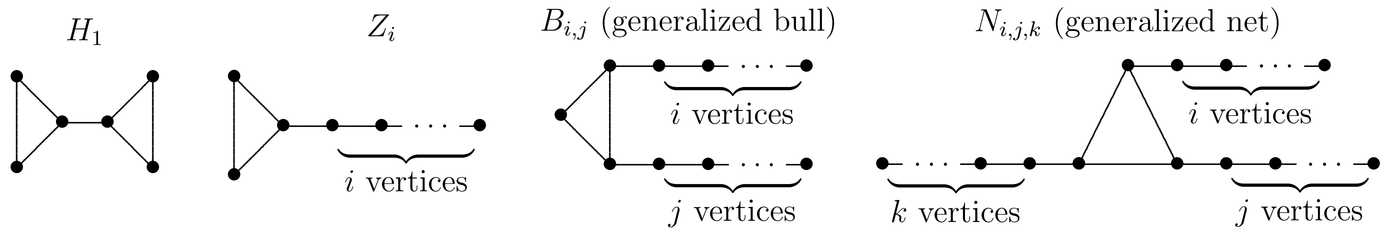
If G is a line graph of a graph, then the "multigraph preimage $L_M^{-1}(G)$" and the obvious line graph preimage $L^{-1}(G)$ can be different!

M-closed graphs

Proposition. *Let G be a claw-free graph and let T_1, T_2, T_3 be the graphs shown in the figure. Then G is M-closed if and only if G is a line graph of a multigraph and $L_M^{-1}(G)$ does not contain a subgraph (not necessarily induced) isomorphic to any of the graphs T_1, T_2 or T_3.*



Application: forbidden subgraphs for Hamilton-connectedness



On the positive side:

Theorem **[Shepherd 1991], [Chen,Gould 2000], [Broersma, Faudree, Huck, Trommel, Veldman 2002].**
Let G be a 3-connected CX-free graph, where $X \in \{P_6, Z_3, B_{1,2}, N_{1,1,1}, H_1\}$. Then G is Hamilton connected.

Opposite direction:

Theorem **[Broersma, Faudree, Huck, Trommel, Veldman 2002]**
If X, Y is a pair of connected graphs such that $X, Y \not\simeq P_3$ and every 3-connected XY-free graph is Hamilton connected, then, up to a symmetry, $X = C$ and Y satisfies each of the following conditions:

(a) $\Delta(Y) \leq 3$,

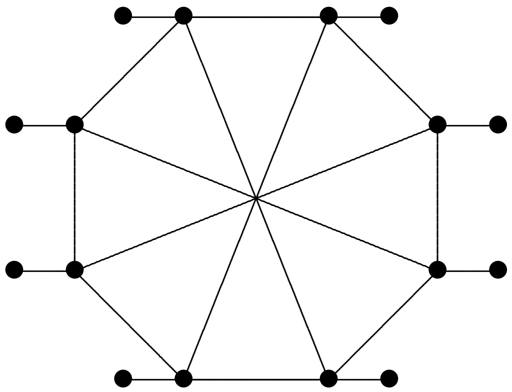
(b) *any longest induced path in Y has at most 9 vertices,*

(c) *Y contains no cycles of length at least 4,*

(d) *the distance between two distinct triangles in Y is either 1 or 3,*

(e) *There are at most two triangles in Y,*

(f) *Y is claw-free.*



$G = L(H)$ is 3-connected and not Hamilton-connected

Thus, the "longest" P_i, Z_i, $B_{i,j}$ and $N_{i,j,k}$ implying Hamilton-connectedness can be:

P_9

Z_6

$B_{i,j}$ for $i + j = 7$

$N_{i,j,k}$ for $i + j + k = 7$

(Recall: known $P_6, Z_3, B_{1,2}, N_{1,1,1}, H_1$).

Using $\mathrm{cl}^M(G)$, the following was proved.

Theorem **[Faudree, Faudree, ZR, Vrána 2009]**
If G is a 3-connected XY-free graph for $X = C$ and $Y = P_8$, $N_{1,1,3}$, or $N_{1,2,2}$, then G is Hamilton-connected.