Five-cycles in 2-factors of cubic graphs

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Snarks

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Oddness – the smallest number of odd cycles in a 2-factor of a given cubic graph.

Two-factors of snarks

The number of 2-factors of a cubic graph:

- conjectured to be exponential in the number of vertices.
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Theorem (Kaiser, Škrekovski)

Every bridgeless cubic graph contains a 2*-factor intersecting all non-trivial edge-cuts of size* 3 *and* 4*.*

No way to intersect all 5-edge-cuts for the Petersen graph.

Short cycles in each 2-factor

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Are there non-trivial snarks with a 5-cycle in each 2-factor?

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Motivation:

Theorem (Král', Máčajová, Mazák, Sereni)

Every cubic graph with a 2-factor without 5-cycles is circularly $7/2$-edge-colourable.

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Cyclic edge-connectivity 5? Open.

Two-factor composed of five-cycles

The Petersen graph P has six 2-factors, each composed of two 5-cycles.

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Problem (Fouquet, Vanherpe)

Is there a cubic graph other then P with a 2-factor containing only 5-cycles?

They constructed such graphs with cyclic edge-connectivity 4 (by using dot product).

Two-factor composed of five-cycles: motivation

Conjecture (Fulkerson)

For every bridgeless cubic graph there is a collection of six perfect matchings such that each edge belongs to exactly two of them.

Tough, widely open. Verified only for a few explicit classes.

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Theorem (Fouquet, Vanherpe 2009)

Let a cubic graph G have a 2-factor F containing only 5-cycles. Let H be the 5-regular multigraph obtained from G by contracting F. If H is 5-edge-colourable then the Fulkerson conjecture holds for G.

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- superposition: looks promising.

Superposition

Theorem (Kochol)

If we replace each vertex of a snark G by a supervertex and each edge by a superedge, then the resulting graph is a snark.

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- Supervertex: no problem.
- Superedge: can be obtained from any snark by removing two non-adjacent vertices.
- We need a suitable cyclically 5-edge-connected snark on $10k + 2$ vertices.

Thank you for your attention.