Forbidden graphs for a graph G with the index $\rho(G^2) < 4$.

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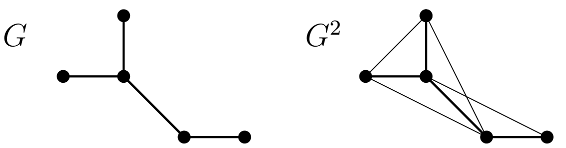
Let $G = (X, Y)$ be an undirected graph without loops and multiple edges. ***The second power*** (or ***square*** of G) is the graph $G^2 = (X, E')$ with the same vertex set X and in which mutually different vertices are adjacent if and only if there is at least one path of length 1 or 2 in G between them.



The characteristic polynomial of the adjacency matrix A of a graph G is called the ***characteristic polynomial*** of G and the eigenvalues and the spectrum of A are called the ***eigenvalues*** and the ***spectrum of G***. The greatest eigenvalue of G is called the ***index*** of G.

Theorem 1. (4) The maximal real eigenvalue r' of every principal submatrix (of an order less than n) of a non/negative matrix A (of an order n) does not exceeded the maximal real eigenvalues r of A. If A is irreducible then $r' < r$ always holds. If A is reducible then $r' = r$ for at least one principal submatrix.

Theorem 2. (1) The increase of any element of a non-negative matrix A does not decrease the maximal real eigenvalue. The maximal real eigenvalue increases strictly if A is an irreducible matrix

Remark 1. Theorems 1. and 2. state that in a (strongly) connected multi-(di)graph G every subgraph (not equal G) has the index smaller than the index of G.

Theorem 3. (7) For the index ρ of a graph $G = (X, E),\ X = \{v_1, v_2, \dots, v_n\}$ it holds

$$\sqrt{\sum_{i=1}^{n} \frac{d_G^2(v_i)}{n}} \le \rho \le \min\left\{\max_i d_G(v_i); \sqrt{\sum_{i=1}^{n} d_G(v_i)}\right\} \tag{1}$$

Corollary 1. (7) Let G contains a vertex v with $d_G(v) \ge 4$ or a circuit C^n of the length $n \ge 5$. Then

$$\rho(G^2) \ge 4. \tag{2}$$

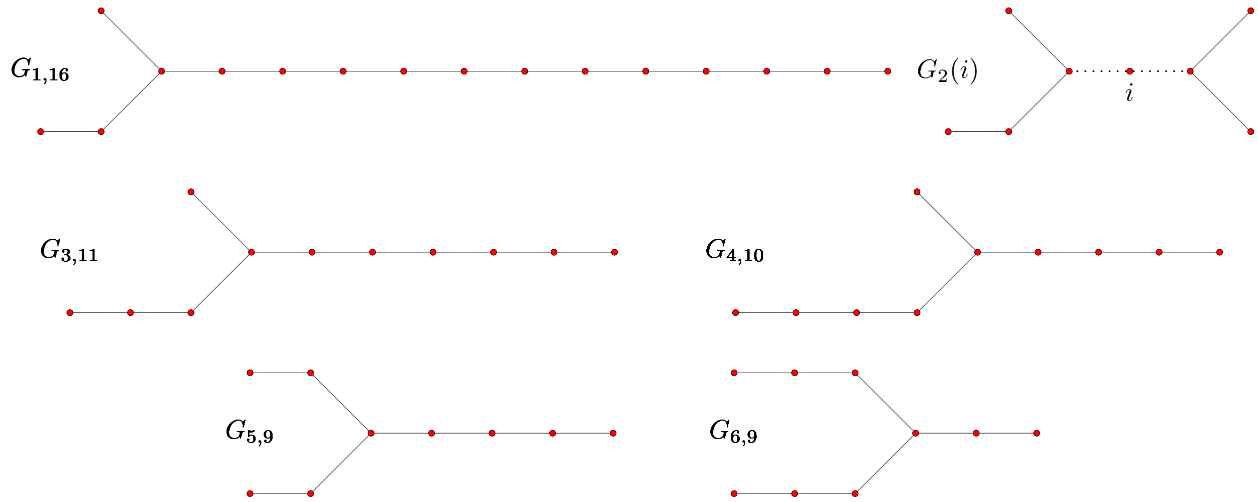
Corollary 2. (7) Let G contains as a subgraph a tree with at least 5 pendant vertices. Then

$$\rho(G^2) > 4. \tag{3}$$

The important problem of the theory of spectra of graphs is a characterization of classes of graphs in term of spectra of graphs. In this paper we will find the system of forbidden subgraphs of the graph G to have the index of its second power $\rho(G^2) < 4$.

1. Main results

Lemma 1. Let G contains as a subgraph at least one of the following graphs (see the figure): $G_{1,16}$ on 16 vertices; $G_2(i)$, $i \geq 0$ on $7+i$ vertices; $G_{3,11}$ on 11 vertices; $G_{4,10}$ on 10 vertices; $G_{5,9}$ on 9 vertices; $G_{6,9}$ on 9 vertices.



Then

$$\rho(G^2) > 4. \tag{4}$$

Lemma 2. Let G contains as a subgraph a block (not necessarily a block of G) on at least 5 vertices or two blocks connected by a path, one of these blocks contains at least 4 vertices. Then

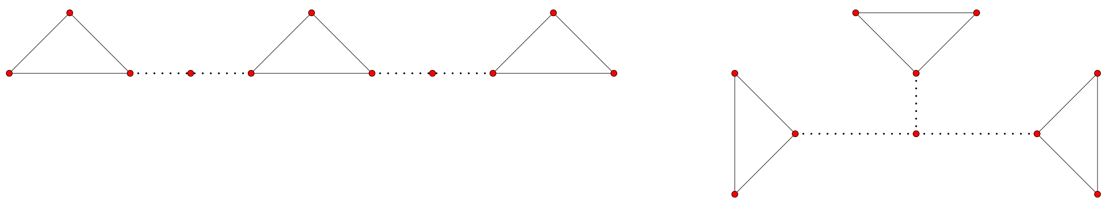
$$\rho(G^2) \geq 4. \tag{5}$$

Proof.
If G contains as a subgraph a block on 5 vertices it has as a subgraph either a circuit $C_k,\ k \geq 5$ or $K_{2,3}$. As the index $\rho(C_k^2 = \rho(K_{2,3})$ the assertion follows from Remark 1. In the second case G has either a vertex of a degree at least 4 and the assertion follows from Corollary 1. or it has as its subgraph one of the following graphs and the assertion follows from Lemma 1.

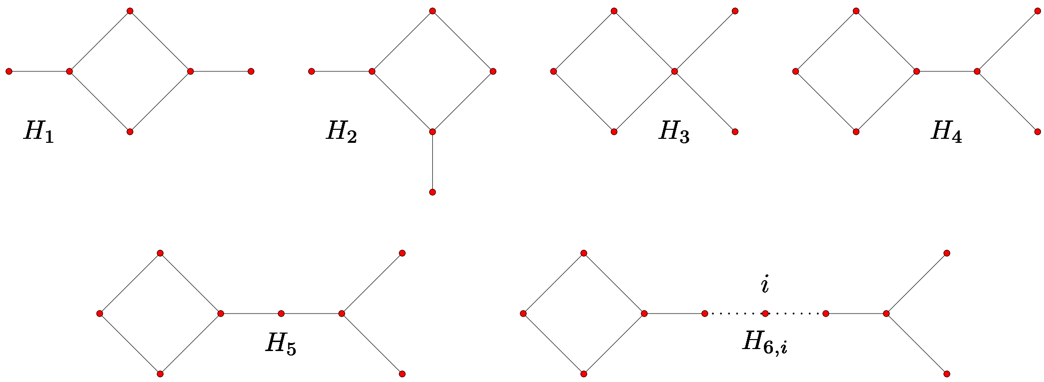
Lemma 3. Let a connected graph G contains as a subgraph at least three blocks or a block on 4 vertices with at least two pendant vertices. Then

$$\rho(G^2) \geq 4. \tag{6}$$

Proof.
Let G contains as a subgraph at least three blocks. With regard to Corollary 1. and Lemma 2. there remain the cases, when G contains as its subgraph one of the following graphs:



and the assertion follows from Corollary 2. Let a connected graph G contains as a subgraph a block on 4 vertices with at least two pendant vertices. Then there remain the cases, when G contains as its subgraph one of the following graphs:



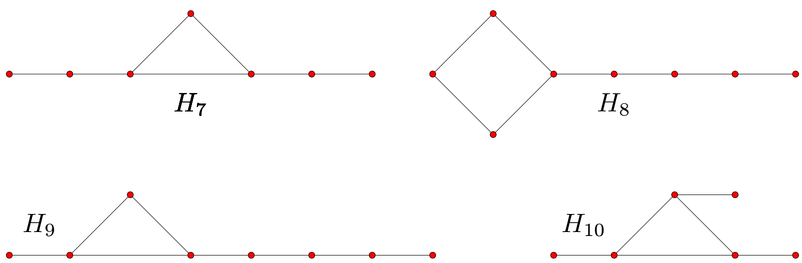
According to Theorem 3. we obtain

$$\rho(H_1^2) \geq \sqrt{16+\frac{4}{6}};\ \rho(H_2^2) \geq \sqrt{16+\frac{4}{6}};\ \rho(H_3^2) \geq \sqrt{16+\frac{4}{6}};\ \rho(H_4^2) \geq \sqrt{16+\frac{8}{7}};$$

$$\rho(H_5^2) \geq \sqrt{16+\frac{8}{8}};\ \rho(H_{6,i}^2) \geq \sqrt{16+\frac{6}{10+i}}.$$

so the assertion follows from Remark 1.

Lemma 4. Let a connected graph G contains as a subgraph one of the following graphs:

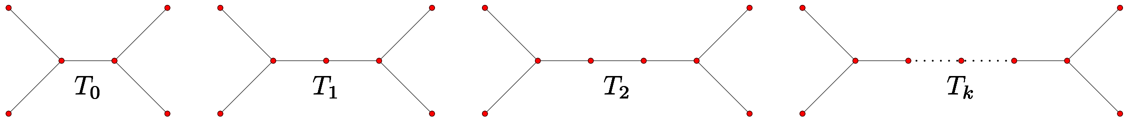


Then

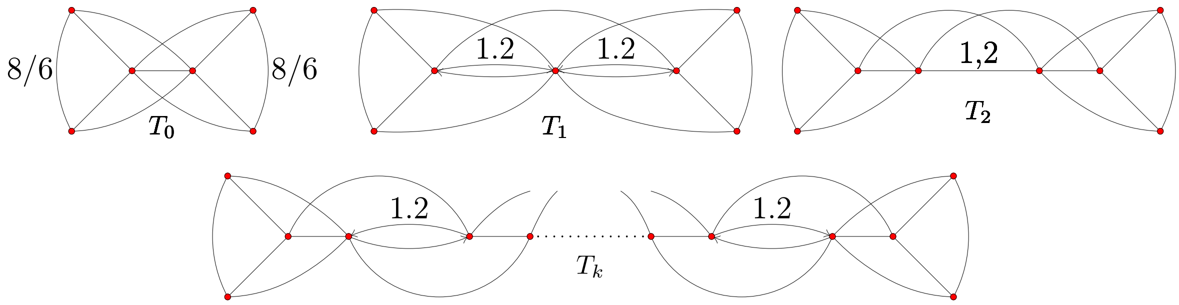
$$\rho(G^2) > 4.$$

Proof.
It is easy to calculate that $\rho(H_i^2) < 4,\ \ i \in \langle 7, 10 \rangle$.

Lemma 5. For the following graphs T_i it holds $\rho(T_i^2) < 4$.

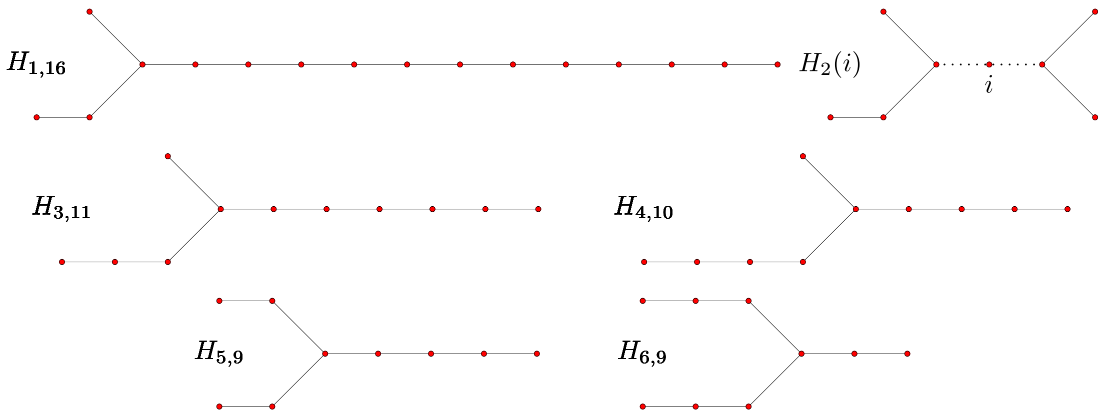


Proof.
The following edge labelled graphs T_i' have the eigenvalue $\rho = 4$ for the corresponding eigenvectors $x = (6, 6, 8, 10, \ldots, 10, 8, 6, 6)$. The unlabelled edges have the weight 1. Then the assertion follows from Remark 1.

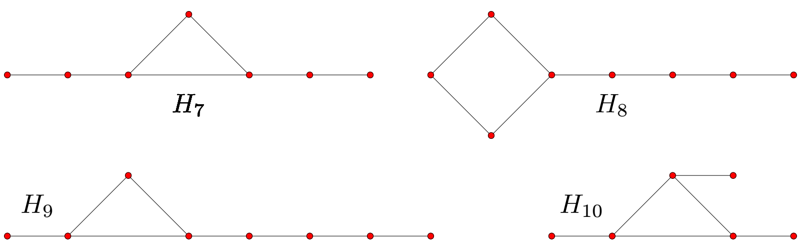


Theorem 4. For a graph G it holds $\rho(G^2) < 4$ if and only if it satisfies the following conditions:

1. G does not contain a vertex v with $d_G(v) \geq 4$ or a circuit C^n of the length $n \geq 5$.
2. G does not contain as a subgraph a tree with at least 5 pendant vertices.
3. G does not contain as a subgraph one of the following graphs: $H_{1,16}$ on 16 vertices; $H_2(i)$, $i \geq 0$ on $7 + i$ vertices; $H_{3,11}$ on 11 vertices; $H_{4,10}$ on 10 vertices; $H_{5,9}$ on 9 vertices; $H_{6,9}$ on 9 vertices.



4. G does not contain as a subgraph a block (not necessarily a block of G) on at least 5 vertices or two blocks connected by a path, one of these blocks contains at least 4 vertices.
5. G does not contain as a subgraph at least three blocks or a block on 4 vertices with at least two pendant vertices.
6. G does not contain as a subgraph one of the following graphs:



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