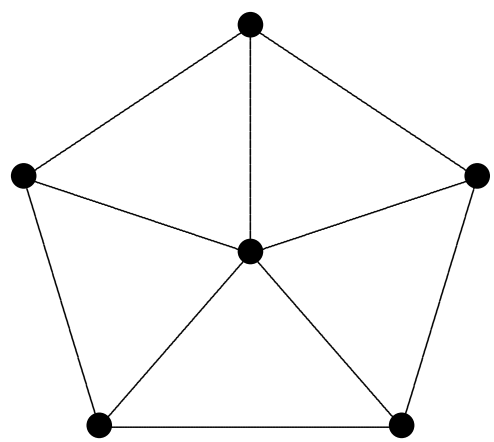
A closure for Hamilton-connectedness in claw-free graphs

Petr Vrána, Roman Kužel
Department of Mathematics
University of West Bohemia
Plzeň, Czech Republic

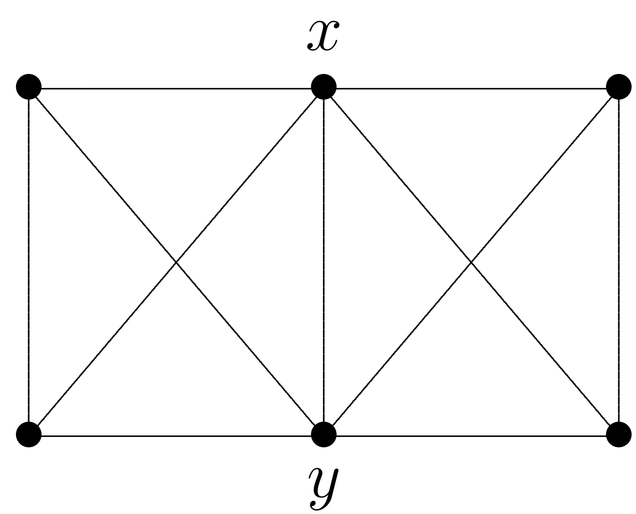
Graph $G = (V(G), E(G))$: simple loopless finite undirected

Hamiltonian (x, y)*-path* in G: an (x, y)-path passing through all vertices

A graph G is *Hamilton-connected* if G has a hamiltonian (x, y)-path for all pairs of vertices $x, y \in V(G)$



A Hamilton-connected graph



This graph is not Hamilton-connected

Observation. *A Hamilton-connected graph must be 3-connected.*

Theorem [Li et al.]. *Let H be a multigraph with $|E(H)| \geq 3$. Then $G = L(H)$ is Hamilton-connected if and only if for any pair of edges $e_1, e_2 \in E(H)$, H has a internally dominating (e_1, e_2)-trail ((e_1, e_2)-IDT).*

Theorem [Ryjáček, P.V.]. *Let G be a connected claw-free graph and let $\mathrm{cl}^M(G)$ be the M-closure of G. Then*
(i) $\mathrm{cl}^M(G)$ is uniquely determined,
(ii) there is a multigraph H such that $\mathrm{cl}^M(G) = L(H)$,
(iii) $\mathrm{cl}^M(G)$ is Hamilton-connected if and only if G is Hamilton-connected

Theorem . *Let G be a connected line graph of a multigraph. Then there is, up to an isomorphism, a uniquely determined multigraph $H = L_M^{-1}(G)$ such that a vertex $e \in V(G)$ is simplicial in G if and only if the corresponding edge $e \in E(H)$ is a pendant edge in H.*

Theorem . *Let G be a claw-free graph. Then there is a graph G^M such that*

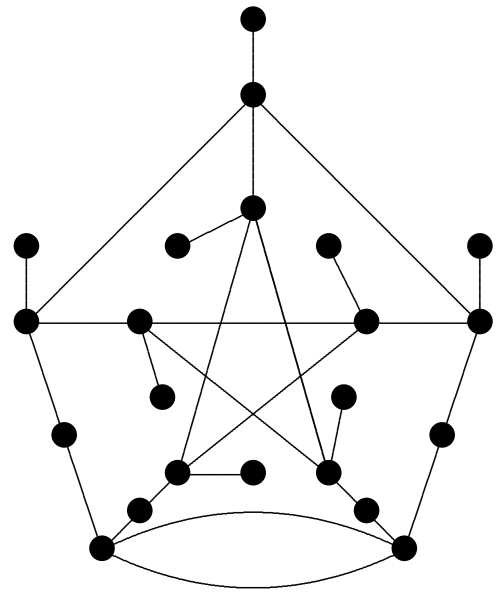
(i) there is a sequence of vertices $x_1, \ldots, x_{k-1} \in V(G)$ and graphs $G_1, \ldots, G_k = G^M$ such that $G_1 = G$, $G_{i+1} = G^*_{x_i}$, $i = 1, \ldots, k-1$,

(ii) G is Hamilton-connected if and only if G^M is Hamilton-connected,

(iii) there is a multigraph H such that

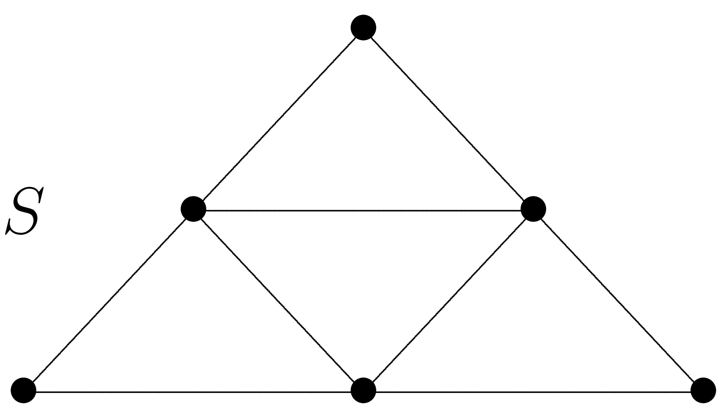
- *(α) $G^M = L(H)$ and H contains no pair of triangles with a common edge and no multiedge of multiplicity more than 2,*
- *(β) H contains either no multiple edge and at most two triangles, or no triangle and at most one multiple edge,*
- *(γ) if H contains a triangle T, then H has an (e, f)-IDT for any $e, f \in E(H)$ with $\{e, f\} \cap E(T) = \emptyset$,*
- *(δ) if H contains a multiple edge ef, then (e, f) is the only pair of edges for which H has no (e, f)-IDT.*

H



$L(H)$ is not Hamilton-connected.

S

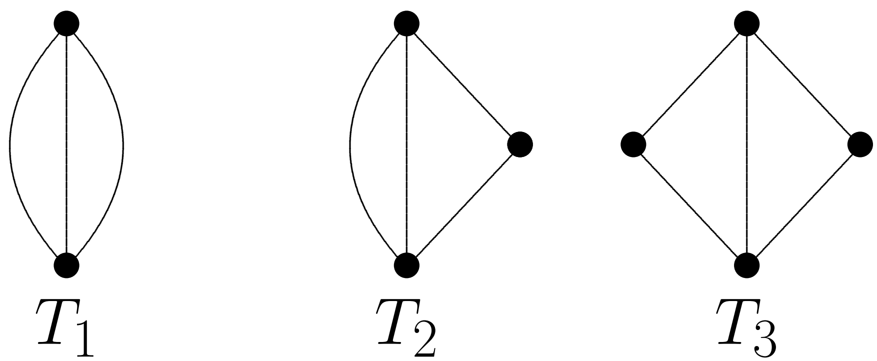


Theorem [Brandt et al.]. *Let x be an eligible vertex of a claw-free graph G, G_x^* the local completion of G at x, and a, b two distinct vertices of G. Then for every longest a, b-path $P'(a, b)$ in G_x^* there is a path P in G such that $V(P) = V(P')$ and P admits at least one of a, b as an endvertex. Moreover, there is an a, b-path $P(a, b)$ in G such that $V(P) = V(P')$ except perhaps in each of the following two situations (up to symmetry between a and b):*

(i) There is an induced subgraph $H \subset G$ isomorphic to the graph S such that both a and x are vertices of degree 4 in H. In this case G contains a path P_b such that b is an endvertex of P and $V(P_b) = V(P')$.

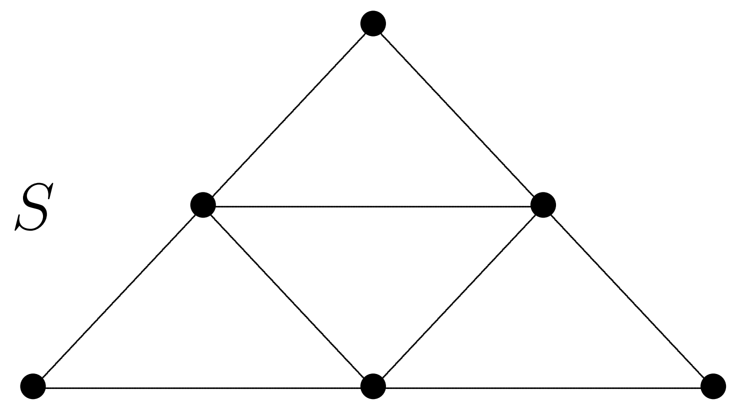
(ii) $x = a$ and $ab \in E(G)$. In this case there is always both a path P_a in G with endvertex a and with $V(P_a) = V(P')$ and a path P_b in G with endvertex b and with $V(P_b) = V(P')$.

Theorem . *Let G be a claw-free graph. Then G is M-closed if and only if G is a line graph of a multigraph and $L_M^{-1}(G)$ does not contain a subgraph (not necessarily induced) isomorphic to any of the graphs T_1, T_2 or T_3.*



Let H be a multigraph with at least three triangles T_1, T_2, T_3 such that there is no IDT between edges e, f.

At least one triangle does not contain the edge e and the edge f. In the graph $L(H)$ is an eligible vertex x such that the local completion at x does not change Hamilton connectedness of $L(G)$.



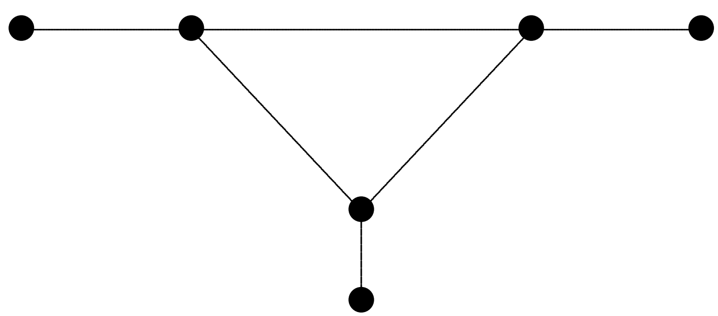
Let H be a multigraph with at least one multiple edge m_1m_2 such that there is no DT between the edges e, f. The multiple edge is not in a triangle. If $m_1 \notin \{e, f\}$ we can close m_1 in $L(H)$.

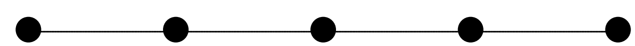
We do not know a polynomial time algorithm for constructing a closure G^M of a graph G

The structure of G^M is very close to that of the closure cl(G) (only at most 2 triangles or one multiedge).

Potential applications:

- forbidden pairs
- degree conditions





Thank you for your attention