NOWHERE-ZERO FLOWS ON BIDIRECTED EULERIAN GRAPHS

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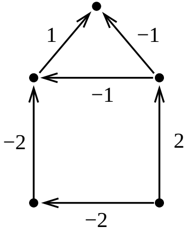
Comenius University, Bratislava

joint work with M. Škoviera

Nowhere-zero flows on graphs

G – graph
$\overrightarrow{G}$ – orientation of G
nowhere-zero k-flow on $\overrightarrow{G}$ is a function
$f : E(G) \to \{\pm 1, \pm 2, ..., \pm(k-1)\}$ such that

$$\sum_{e \in E^+(v)} f(e) = \sum_{e \in E^-(v)} f(e)$$



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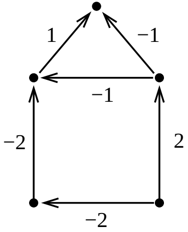

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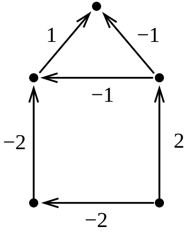

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Theorem (Seymour, 1981)

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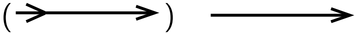
- ordinary edges one half-edge directed from and the other half-edge towards its end-vertex 

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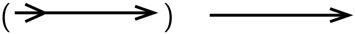
- ordinary edges one half-edge directed from and the other half-edge towards its end-vertex 
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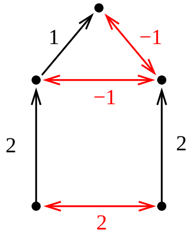

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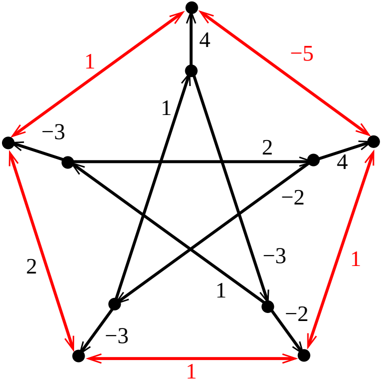
Theorem (EM, Škoviera 2010)

Bouchet's conjecture is true, if it is true for bidirected cubic graphs.

Example: a NZ 6-flow on the Petersen graph



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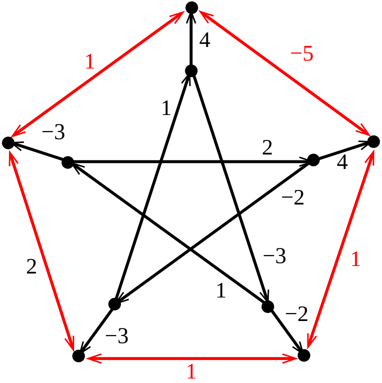

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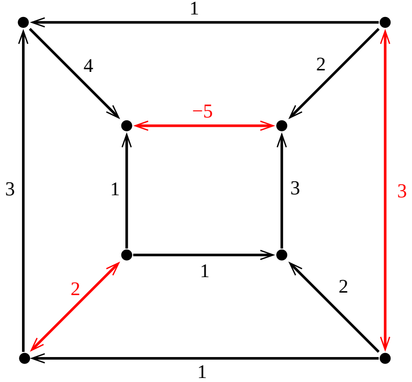

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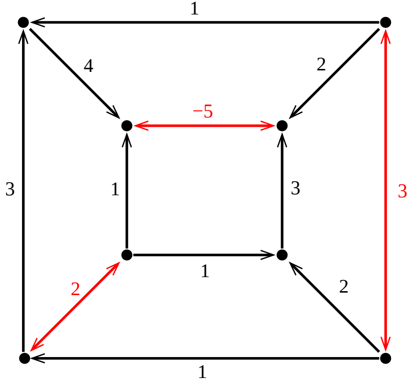

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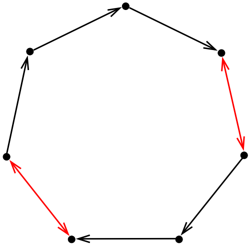
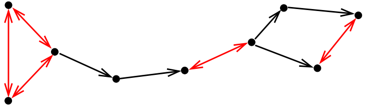
Theorem (Zaslavsky 1982, Bouchet 1983)

A bidirected graph admits a NZ k-flow for some k if its edges can be covered by

- *balanced circuit*
- *pairs of unbalanced circuits connected by a path*





Results

If a bidirected G admits a NZ flow, then

- 216-flow Bouchet (1983)
- 18-flow if G is 4-connected Khelladi (1987)
- 30-flow Zýka (1987)
- 6-flow if G is 6-edge connected Xu, Zhang (2005)
- 4-flow if G is 4-connected Raspaud, Zhu
- 12-flow DeVos (???)

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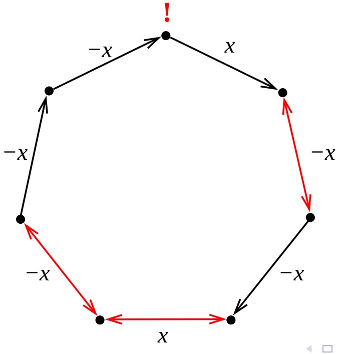

Nowhere-zero 2-flows on bidirected graphs

Theorem (Xu, Zhang, 2005)

Let G be a connected bidirected graph which admits a nowhere-zero flow. Then G has a NZ 2-flow $\iff$ G is eulerian and has even number of broken edges.

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Main result

Theorem (EM, Škoviera, 2010)

Let G be a bidirected eulerian graph which admits a NZ flow. Then G admits a NZ 4-flow.

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- two subcases depending on whether or not A', B', and C' have a vertex in common

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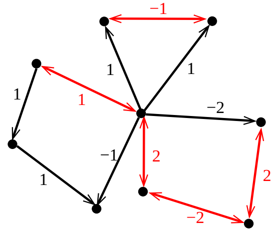
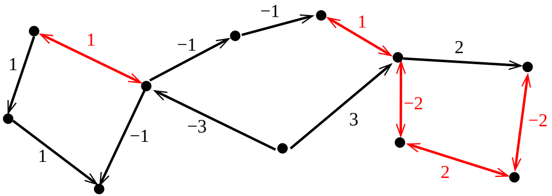


if A', B', and C' do not share a vertex:



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- we combine them into a NZ 4-flow on G

Conjecture

$\Phi(G)$ – flow number of a bidirected graph G: minimal k such that G admits a NZ k-flow

Conjecture (Máčajová, Škoviera, 2010)

Let G be an bidirected eulerian graph with $\Phi(G) = 3$. Then G can be decomposed into three closed trails sharing a vertex, each having an odd number of broken edges.

Enough to prove for 6-regular graphs