# Neighbour-distinguishing index of planar graphs

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# Introduction

$G$ a graph without isolated edges
$C$ a set of colours $\qquad \varphi : E(G) \to C$ a proper colouring
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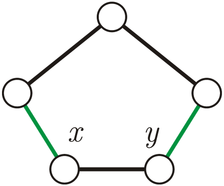
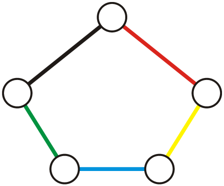



$S_\varphi(x) = S_\varphi(y)$



$\mathrm{ndi}(C_5) = 5$

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Hatami 2005 ... $\Delta(G) > 10^{20} \Rightarrow \mathrm{ndi}(G) \leq \Delta(G) + 300$

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$(g(G) \geq 14 \wedge \Delta(G) = 3) \Rightarrow \mathrm{ndi}(G) \leq \Delta(G) + 1$

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$\downarrow$
a contradiction to Euler's Formula
for an appropriate plane embedding of $G$

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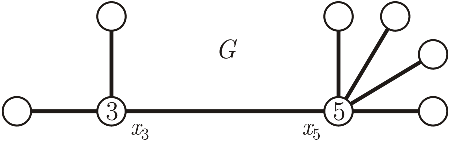
$G$ an ndi-minimal graph with a $(3, 5)$-edge $x_3x_5$



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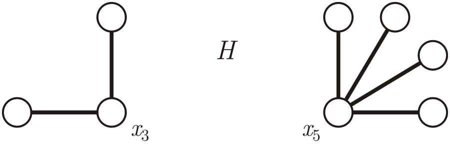


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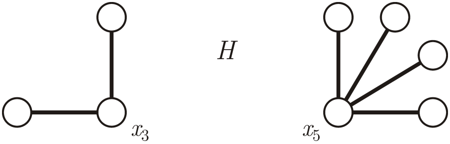


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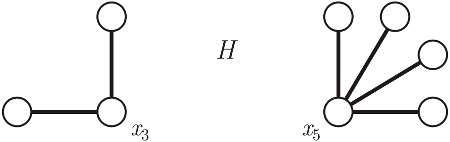


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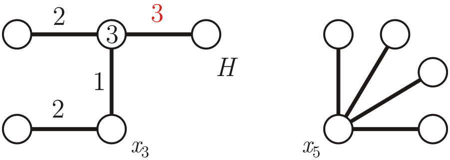


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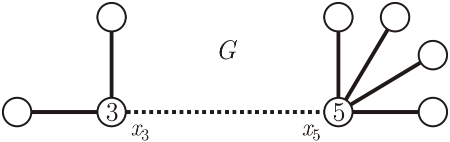


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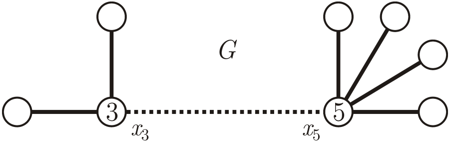


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$$G = g(i, j) = \begin{pmatrix} 11 & 11 & 10 & 10 & 9 \\ 11 & 10 & 10 & 9 & 9 \\ 10 & 10 & 8 & 8 & 8 \\ 10 & 9 & 8 & 7 & 7 \\ 9 & 9 & 8 & 7 & 7 \end{pmatrix}$$

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$c_0(v) = \sum_{w \in N(G)} s(v, w)$

*contribution* of the (oriented) edge $(v, w)$

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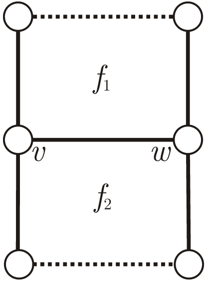

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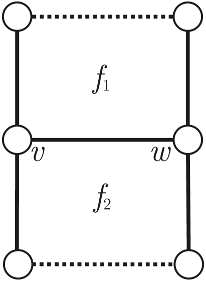


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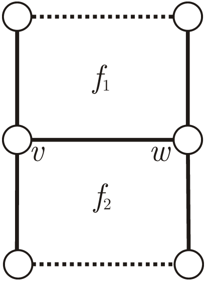


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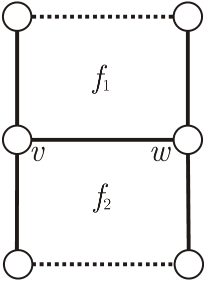


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# Discharging Method

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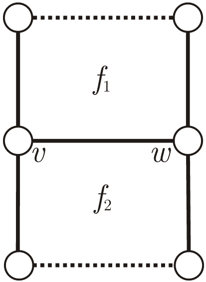


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$\sum_{v \in V} c_2(v) = \sum_{v \in V} c_1(v) = \sum_{v \in V} c_0(v) = 2$
while $c_2(v) \le 0$ for all $v \in V$ ($\leftarrow$ reducible configurations) ↯

# Redistribution Rules, stage 1

positive contributions $\longrightarrow$ $c_0$-*negative* vertices ... $c_0(w) < 0$

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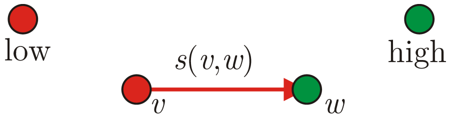

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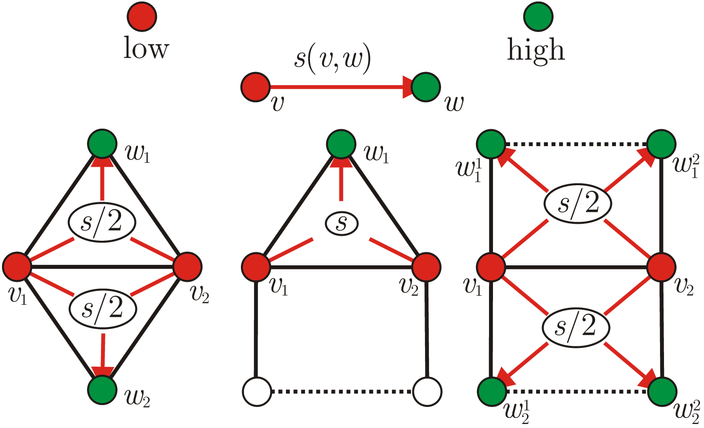


$s(v_1,v_2)=s(v_2,v_1)=s$

# Redistribution Rules, stage 2

a vertex $v$ with $c_1(v) < 0$ ($c_1$-*positive*) $\Rightarrow \deg(v) = 8$,
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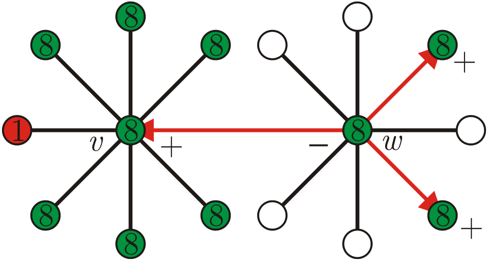

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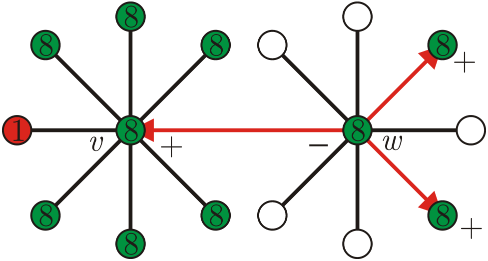


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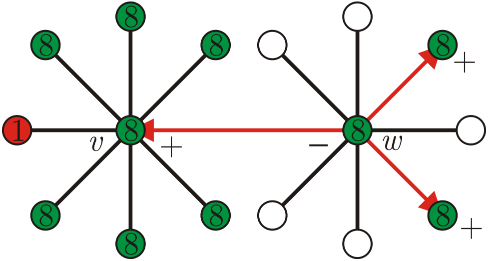


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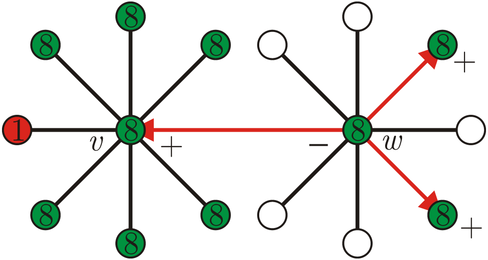


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it can be shown that all vertices are $c_2$-*nonpositive* ↯

Thank you.