Colorings and crossings

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Maps and colors

One of the fundamental motivations in graph coloring is the problem of coloring (political) map by as few colors as possible.

4 colors - necessary, but are enough?

4 color problem can be translated into (planar) graphs coloring

Drawing of a graph in the plane

Let $G = (V, E)$ be a graph.

- V corresponds to set of points in the plane (bijection)
- every edge corresponds to an arc
- arcs do not have a common points (except vertices)
- arcs join corresponding vertices

Graph G is planar if it can be drawn in the plane.

Graph colorings

Let $G = (V, E)$ be a graph and C set of colors.

- *coloring* is a mapping $c : V \to C$.
- coloring is *proper* if adjacent vertices have distinct colours
- *chromatic number* $\chi(G)$ is minimum k such that G can be properly colored using k colors.

In what follows, we consider only proper colorings.

Four color theorem

Planar graph characterization:

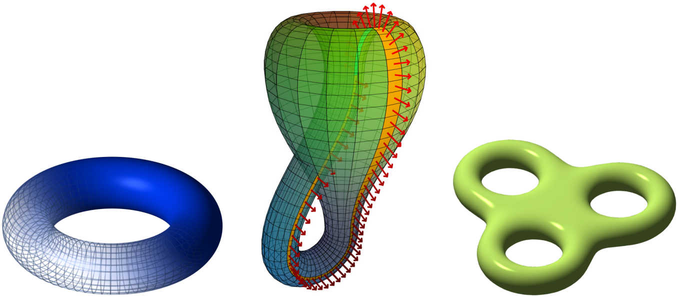
- no $K_{3,3}$ nor K_5 as a minor
- no $K_{3,3}$ nor K_5 as a subdivision

Theorem (Appel, Haken 1989 /
Robertson, Sanders, Seymour, Thomas 1996)

Every planar graph can be colored by at most 4 *colors.*

Other surfaces

Add handles and/or cross caps to the sphere.



Every graph can be drawn on some surface - defines genus of a graph.

Genus and the chromatic number - Heawood formula

What is the smallest c such that every graph G of Euler genus at most g is c-colorable?

$$c \leq \left\lfloor \frac{7 + \sqrt{24g + 1}}{2} \right\rfloor$$

(except the Klein bottle holds with equality)

Planar graphs are the difficult case

k-critical graphs

A graph $G = (V, E)$ is k-critical is $\chi(G) = k$ and for every $x \in V \cup E : \chi(G - x) < k$.

Example: K_n is n-critical

Knowledge of k-critical graphs help with bounding $\chi(G)$.

k-critical graphs on surfaces

How many k-critical graphs are on a given surface?

k	number	author	year
≥ 8	finite	Dirac	1956
7	finite	Thomassen	1994
6	finite	Thomassen	1997
5	infinite	Fisk	1978
4	infinite	Fisk	1978
3	infinite	Fisk	1978

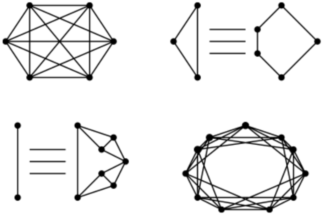
Do we know some of the lists?

6-critical graphs on surfaces

1. projective plane Dirac, 1956

   K_6

2. torus Thomassen, 1994



3. Klein bottle Kawarabayashi, Král', Kynčl and Lidický, 2008 independently Chenette, Postle, Streib, Thomas and Yerger, 2008

   list of 9 graphs

Crossings

Edges in a drawing are allowed to cross.
(at most two in one point)

Let G be a graph. Its crossing number $\mathrm{cr}(G)$ is the minimum crossings needed for drawing G.

There are other notions of crossing number.

Cluster of the crossining is formed by vertices of crossed edges.

Close clusters ...

Observation

If all clusters have a common vertex, then $\chi(G) \leq 5$*.*

Let $G = (V, E)$ be a graph. An independent set $I \subseteq V$ is a stable crossing cover if $G - I$ is planar.

Observation

If G has stable crossing cover, then $\chi(G) \leq 5$*.*

... or distant clusters?

Theorem (Král', Stacho, 2008)

If clusters of all crossings are disjoint, then $\chi(G) \leq 5$.

Albertson conjecture

Conjecture (Albertson)

$$\mathrm{cr}(K_n) = \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor$$

Verified for $n \leq 10$.

Crossing number and coloring

What is the smallest c such that every graph G of $\mathrm{cr}(G) \leq k$ is c-colorable?
Denote the answer by $f(k)$

- $f(0) = 4$
- $f(1) = 5$ [K_5]
- $f(2) = 5$ [Oporowski and Zhao, 2008]
- $f(3) = 6$ [K_6]
- $f(6) = 6$ [Albertson, Heneehan, McDonough and Wise]
- $f(k) = O(k^{1/4})$ [Schaefer] tight because of K_n

Albertson conjecture

Conjecture (Hajós)

If $\chi(G) \geq n$ then G contains subdivision of K_n.

False for $n \geq 7$.

Conjecture (Albertson)

If $\chi(G) \geq n$ then $\mathrm{cr}(G) \geq \mathrm{cr}(K_n)$.

- $n = 5$ equivalent to the 4-color theorem
- $n = 6$ implied by results of Oporowski and Zhao, 2008
- Albertson, Cranston and Fox (2009) proved for $n \leq 12$
- Barát, Tóth (2010) proved for $n \leq 16$

Crossing number and 6-critical graphs

Theorem (Oporowski and Zhao, 2008)

If $\mathrm{cr}(G) \leq 3$ *and* $\omega(G) \leq 5$ *then G is 5 colorable.*
The only 6-critical graph with $\mathrm{cr}(G) \leq 3$ *is* K_6*.*

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Three edges

Theorem (Oporowski and Zhao, 2008)
The only 6-critical graph with $\mathrm{cr}(G) \leq 3$ *is* K_6*.*

Theorem (EHLP)
The only 6-critical graph which is planar after removing three edges is K_6*.*
If G *is planar after removing three edges and* $\omega(G) \leq 5$ *then* G *is 5 colorable.*

Four crossings

Theorem (EHLP)

The only 6-critical graph with $\mathrm{cr}(G) \leq 4$ *is* K_6.
If $\mathrm{cr}(G) \leq 4$ *and* $\omega(G) \leq 5$ *then* G *is 5 colorable.*

Proof ideas:

- $\mathrm{cr}(G) \leq 4 \Rightarrow G$ contains (at least four) 5-vertices
- stable crossing cover
- small separations
- contractions along non-adjacent neighbors of a 5-vertex
- analog of Kempe chains

Proof ideas - separations

Minimal counterexample G (6-critical, $\mathrm{cr}(G) \leq 4$, $G \not\simeq K_6$) has

- no separating regular 3-cycle
- no separating 3-cycle with at most one of its edges crossed and at most one crossing inside
- no separating non-crossed 4-cycle with a chord outside and no crossing inside

Proof ideas - contraction

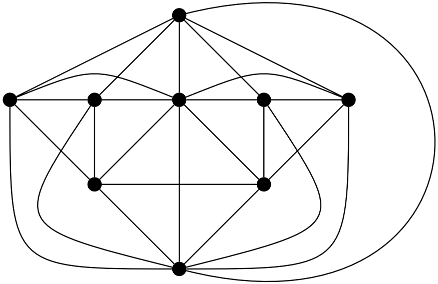
Lemma

Let G be a graph and v its 5-vertex and let x and y be two non-adjacent neighbors of v. If $(G - v)/\{x, y\}$ *is* 5*-colorable then so is G.*

5 crossings - counterexample

Theorem (EHLP + Dvořák)

There exists a 6-critical graph with $\mathrm{cr}(G) = 5$ *different from* K_6.



What next?

Problem

List all 6-critical graphs with 5 crossings.

Problem

Determine $f(k)$ *for* $k \geq 7$.

Problem

Is the number of 5-critical graph of crossing number k bounded?

Problem

Are graphs of $cr(G) = 2$ *5-choosable?*