Automorphisms in the degree-diameter problem for vertex transitive digraphs

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Introduction

Motivation : design of (symmetric) interconnection networks

Degree-diameter problem for graphs and digraphs:
to determine the largest number of vertices in a graph (digraph) of given maximum degree and diameter

Directed case

- (Δ, D)-problem: the construction of digraphs with an order as large as possible for a given degree and diameter
- The directed Moore Bound:

$$n(\Delta, D) \leq 1 + \Delta + \Delta^2 + \ldots + \Delta^D$$

- Vertex-transitive digraphs

Constructions:

- Faber-Moore-Chen
- Comellas and Fiol
- Gómez

Construction of Comellas and Fiol

Γ - a digraph of out-degree Δ, with vertex set V and arc set E

- t, k - positive integers
- Vertices of $CF(\Gamma, k)$ are $(j|p_0p_1 \ldots p_{k-1})$ with $j \in \mathbb{Z}_{kt}$ and $p_i \in V = \mathbb{Z}_n$
- Adjacencies are given by:

  $$(j|p_0 \ldots p_{j-1}\ u\ p_{j+1} \ldots p_{k-1}) \rightarrow (j{+}1|p_0 \ldots p_{j-1}\ v\ p_{j+1} \ldots p_{k-1}),$$

  where $(u, v) \in E$
- Order: $kt.n^k$ and diameter at most $k(s+t) - 1$.

AIM

- analyze the Comellas and Fiol digraphs
- natural automorphisms: if $(g_0, \dots, g_{k-1}; s) \in [\mathrm{Aut}(\Gamma)]^k \rtimes \mathbb{Z}_{kt}$

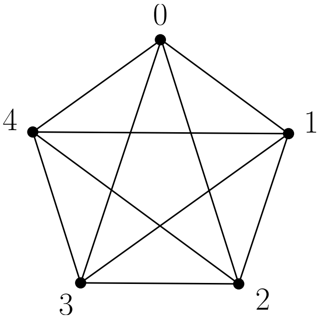
$$\alpha(j|p_0, \dots, p_{k-1}) = (j - s|g_0(p_s) \dots g_{k-1}(p_{s+k-1}))$$

- determine the full group of automorphism of these digraphs for wide class of input digraphs
- examples with bigger automorphism group

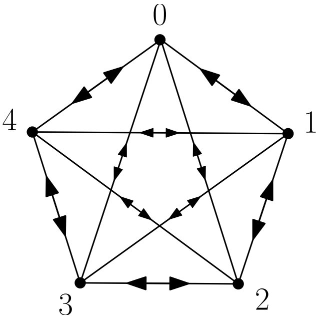
Preliminaries

- a digraph is vertex-transitive if for every pair of vertices u and v there exists an automorphism of Γ that carries u to v.
- an alternating walk from u to v in Γ is a walk that contains no directed sub-walk of length two.
- digraph Γ is r-alternately reachable if there is a number r such that for every pair of vertices u and v there exist two alternating walks from u to v such that
    - one begins with an arc from u
    - second begins with an arc terminating at u.
- r'-alternately reachable $\Rightarrow$ r-alternately reachable for $\forall r \geq r'$

Example 1



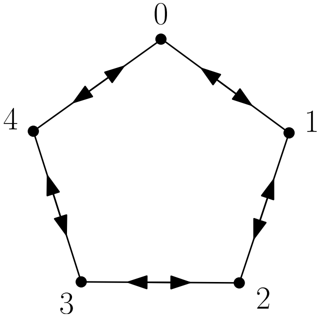
2-reachable



2-alternately reachable

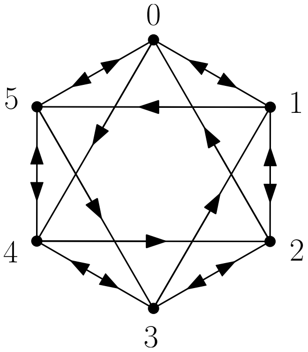
Example 2

- $n - odd$
- n-alternately reachable



Example 3

- *n* − *even*
- arcs: $(i, i+1)$ and $(i, i-1)$ and $(i, i+2m)$ for some m is $(2m + 1 + \frac{n}{2})$ - alternately reachable



Results

Theorem

If the input digraph Γ *is* r*-alternately reachable and vertex-transitive, then the order of the automorphism group of the Comellas-Fiol digraph* $CF(\Gamma, k)$ *where* $t = 1$ *is*
$|Aut(CF(\Gamma, k))| = k|Aut(\Gamma)|^k$.
Consequently, $Aut(CF(\Gamma, k)) \cong [Aut(\Gamma)]^k \rtimes \mathbb{Z}_k$.

Corollary

The automorphism group of the Comellas-Fiol output digraphs $CF(\Gamma, k)$*, where* Γ *is as in Example 1, 2 or 3 and* $t = 1$*, is isomorphic to* $[Aut(\Gamma)]^k \rtimes \mathbb{Z}_k$.

Lemma

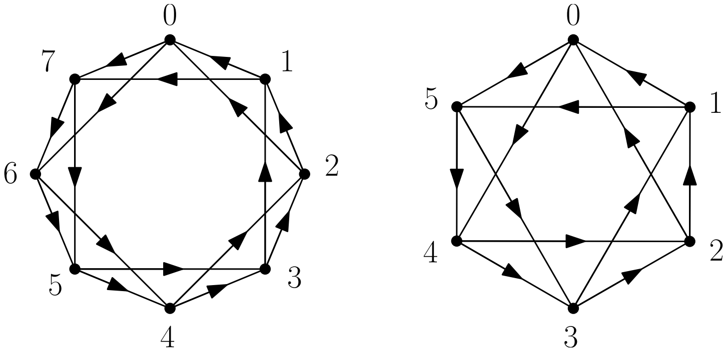
$|\mathrm{Aut}(CF(\Gamma, k))| = |V(CF(\Gamma, k))| \cdot |\mathrm{Stab}\,(v)|$

Lemma

Let the input digraph Γ be r-alternately reachable and vertex-transitive with the vertex set $V = \mathbb{Z}_n$. Let σ be an automorphism of $CF(\Gamma, k)$: $\sigma(0|0\ldots0) = (0|0\ldots0)$. Then

- *σ preserves the set $[(j|z_0 \ldots z_{i-1} 0_i z_{i+1} \ldots z_{k-1})]$, where $i, j \in \mathbb{Z}_k$ and $\forall l$, $z_l \in \mathbb{Z}_n$*
- *the automorphism σ must be of type $\sigma(i|z_0 \ldots z_{k-1}) = (i|\varphi_0(z_0) \ldots \varphi_{k-1}(z_{k-1}))$ where $\varphi_j : \mathbb{Z}_n \to \mathbb{Z}_n$ is a bijection such that $\varphi_j(0) = 0$ for $\forall i, j \in \mathbb{Z}_k$.*
- *the automorphism σ must be of type $\sigma(i|z_0 \ldots z_{k-1}) = (i|\varphi_0(z_0) \ldots \varphi_{k-1}(z_{k-1}))$ where $\varphi_j : \mathbb{Z}_n \to \mathbb{Z}_n$ is an automorphism of Γ such that $\varphi_j(0) = 0$*

Bigger automorphism group



Stab(v)	1	4	1	4	1	8	1
k	2	2	2	2	3	3	3
t	1	2	3	4	1	2	3

Conclusion

- the full automorphism group of Comellas-Fiol digraphs in the case when $t = 1$ and the input digraph is r-alternately reachable.
- the automorphism group H of the output digraph may be bigger; in particular, we may have $|H| > kt.|\mathrm{Aut}(\Gamma)|^k$.
- it would be interesting to find the automorphism group of the output digraph under weaker restrictions on the input digraphs, or for $t \geq 2$.

Thank you for your attention.